Maths Summer School

Dr David Hodge, School of Computing, Engineering and Built Environment

August 2022

Contents

Chapter 1

Introduction

This document contains an overview of the Maths Summer School, each topic will appear as a separate chapter. A full set of notes in PDF format can be obtained here: PDF version. This PDF version will not contain certain embedded elements (videos and interactives). The PDF should hopefully indicate where content such as this has been omitted.

To help draw your attention to various key elements in the notes you will find some colour-coded boxes. Everything in boxes is likely to be particularly useful to read.

Though if interested you will find the content is approximately:

Laws and Rules

Examples

Warnings

Comments

Definitions

Practice questions

The notes also contain superscripts like this one[1]. Their purpose is generally to provide further information which while strictly not necessary[2] might prove useful if you want a further explanation of a particular point. Mostly they provide a little more context or example to clarify something in a sentence.

It is **strongly recommended** that you attempt the questions embedded in the notes **as you go along**, there are also specific sections at the end of the notes with many more examples to try later for revision purposes (see Sections 6, 7 and 8).

[1] you found it!
[2] hence it's hidden away

Chapter 2

Summary of chapters

These notes contain three chapters: Algebra, Vectors and Matrices. Certain elements in the vectors chapter are useful for understanding the matrix chapter, and the general algebra rules apply to most areas of maths.

2.1 Algebra

This chapter will introduce you to a range of standard topics in *Algebra*, from working with mathematical operations, to solving a variety of different types of standard equations. It will no doubt start at a level you have seen before, but will hopefully also provide some more in depth discussions about topics you have seen before, and help build up your toolkit for tackling algebra.

2.2 Vectors

This chapter will introduce you to what may be a new type of mathematical object, a *vector*. They provide a way to store more than just a single number behind an object we might algebraically just call v. Vectors will be a way to store both length and direction inside one object. This chapter will introduce you to the standard algebra possible on such objects, a variety of different (but equivalent) notations along with applications of their usage in two and three dimensions.

2.3 Introduction to Matrices

Matrices[1] are another new type of mathematical object, this time grids of numbers. You will again check how to do basic algebra on matrices, before learning how matrix multiplication works. A wide range of adjectives used with matrices will be discussed, before introducing the idea of a matrix determinant and inverse. The chapter ends with direct applications of inverses to solving systems of linear equations.

[1] the singular is matrix

Chapter 3

Algebra

Algebra is all about manipulating formulae, moving terms around, adding or subtracting things from both sides of an equation, and other related techniques.

However, the **key** feature of algebra is the use of letters to represent unknown values. This makes it more advanced than just standard *arithmetic*.

3.1 Operations and expressions

This section is designed as a reminder of certain basic techniques; namely using the basic operators ($+, -, \div, \times$) and a reminder of powers (also called indices).

3.1.1 Basic Operations of Arithmetic

We might talk of the addition operator $+$, subtraction operator $-$, division operator $\div$ or $/$, and multiplication operator $\times$. By *operation* all that is meant is an agreed procedure (in our cases they will all involve two numbers). We also sometimes need brackets to help clarify unambiguously the order in which the operations should be performed. There is a fifth *operator* which doesn't have a symbol, namely *exponentiation* (just a fancy word for 'taking powers'). On most calculators this fifth operator is represented by a button that says x^y or $x^\square$ or $\square^\square$.

An *expression* is the name we will give to any collection of maths symbols intended to have a value or represent something.

The order in which we perform a calculation is very important, otherwise we may achieve a different answer to someone else.

A point we will keep returning to is the importance of agreed conventions and meanings so that our written maths, equations and solutions are unambiguous.

Take, for example,

$$9 + 4 \times 3. \tag{3.1}$$

The agreed convention for such an *expression* is to perform the multiplication ($\times$) before the addition ($+$). Many of you will have seen an acronym like BODMAS or BIDMAS to help learn this ordering (*First to Last*).

BODMAS stands for…
Brackets **O**rders **D**ivision **M**ultiplication **A**ddition **S**ubtraction
where **O**rders refers to powers (i.e. exponentiation)

So for the above expression, (3.1), we need to calculate $4 \times 3 = 12$ and add this to 9, resulting in an answer of 21.[1]

> Beware, however, that it isn't quite as easy as just learning this one sequence. Luckily you can always just insert brackets to make it obvious.

Unfortunately it's not always exactly this simple when an expression contains multiple similar or opposite symbols. Division and Multiplication come as a pair and are opposites of each other[2], as do Addition and Subtraction which are also opposites of each other[3]. If we have an expression made of just $+$ and $-$ signs then we should work left-to-right (but still do brackets or powers first as usual). The same applies for $\times$ and $\div$ though this is rarer and we often just use brackets.

Here are a few examples of these complicated cases:

- $3 \div 6 \times 4$ means $0.5 \times 4 = 2$ (left-to-right)
- $20 \div (5 \times 2)$ means $20 \div 10 = 2$ (brackets first)
- $2 - 6 - (4 - 3)$ means $2 - 6 - 1 = -5$ (brackets first)
- $-4 + 3 - 6 + 8 = 1$ (left-to-right)

All of these could be made abundantly clear with the addition of extra brackets (sometimes inside each other) but people are often lazy. For example, the first case above could have been written $(3 \div 6) \times 4$ to remove any confusion. Or the final one could have been written $((-4 + 3) - 6) + 8$.

```
#> [Embedded question appears here in html version
#>  the hyperlink is provided below]
```

Link to Numbas on the web

3.1.2 Expanding brackets

The correct procedure for expanding brackets becomes more complicated the more brackets which are multiplied by each other. At its most simple just one number appears outside a bracket, and you just multiply that one number sequentially by every number inside the bracket, i.e.

$$7\,(3 - x + 2y) = 7 \times 3 - 7 \times x + 7 \times 2y = 21 - 7x + 14y$$

Notice that if the factor on the outside is negative then we sometimes will get two minus signs, $--$ which equals $+$. For example,

$$\begin{aligned} -4\,(-2 + 3x - 7y) &= -4 \times (-2) - 4 \times 3 - 4 \times (-7y) \\ &= - - 8 - 12x - -28y \\ &= 8 - 12x + 28y \end{aligned}$$

When a bracket is multiplied by a bracket then every combination product of one term from each needs to be calculated and then all combined. So in general,

$$(A + B)\,(C + D) = A \times C + A \times D + B \times C + B \times D,$$

or for larger brackets

$$(A + B + C)\,(D + E) = AD + AE + BD + BE + CD + CE.$$

(where all terms on the right are products, $AD = A \times D$ etc...)

There are a few ways to learn a systematic procedure to ensure no terms are missed, a demonstration video appears below for a specific typical example. However, the main point to remember is that every combination

[1] You may want to check for yourself that the incorrect left-to-right reading gives a different answer.
[2] multiplying by 2 and dividing by 2 are opposites
[3] adding 3 and subtracting 3 are opposites

of one term from one bracket with one term from the other bracket must be included in the answer[4]. A method known as the *grid method for expanding brackets* is particularly worth looking into for anyone who wants to read further.

```
#> [Video appears here in html version
#>  the hyperlink is provided below]
```

Alternative direct video link:

https://gcu.planetestream.com/View.aspx?id=3449~4u~vz7dULN7

3.1.3 Powers

Powers are also called Indices, or (occasionally) Exponentiation. Powers are fundamentally just a time-saving notation used to save writing out repeated products.

So we write

- a^2 instead of $a \times a$,
- a^3 instead of $a \times a \times a$,
- a^8 instead of $a \times a \times a \times a \times a \times a \times a \times a$,

This notation is frequently used when performing algebra to re-arrange equations, or to solve for some unknown variable. To help with these manipulations there are a few useful 'rules' or 'laws' to learn (really they're just common patterns you can use to simplify quickly).

From this basic definition the three main laws of powers are actually fairly straightforward:

Law 1: For all x and all a, b, even negative or decimal values…

$$x^a \times x^b = x^{a+b} \tag{3.2}$$

Law 2: For all x and all a, b, even negative or decimal values…

$$x^a \div x^b = x^{a-b} \tag{3.3}$$

Law 3: For all x and all m, n, even negative or decimal values…

$$(x^m)^n = x^{m \times n} \tag{3.4}$$

You will have seen these laws before and with repeated use you will no doubt become familiar with them. The intermediate step for anyone not fully familiar with these rules is to at least know that these laws exist, and be able to re-discover them for yourself with a small example.

Here's a very useful example which uses the powers of 3 and 2 to work out all three laws!

$$x^3 \times x^2 = (x \times x \times x) \times (x \times x) = (x \times x \times x \times x \times x) = x^5 \tag{3.5}$$

So, $x^3 \times x^2 = x^5$ tells us the rule for a *product* must be to *add* the powers. Since the only way to get 5 from 2 and 3 is to add them.

The same idea can be used to work out the other laws too:

$$y^3 \div y^2 = (y \times y \times y) \div (y \times y) = \frac{y \times y \times y}{y \times y} = \frac{y}{1} = y \tag{3.6}$$

[4] terms are blobs of letters/numbers separated by plus or minus signs

We used unknown letter y here just to illustrate that the name of the unknown isn't relevant. We see that $y^3 \div y^2 = y^{3-2} = y^1$ so the rule for *division* is to *subtract* the second power from the first. It is perhaps worthy of note that the $\div$ symbol is just a shorthand for writing out a fraction, so $y^3 \div y^2$ is just another way of writing $\frac{y^3}{y^2}$.

Finally, we can again use 3 and 2 as powers to illustrate the third law.

$$\left(x^3\right)^2 = (x \times x \times x) \times (x \times x \times x) = x^6 \tag{3.7}$$

So we notice that $\left(x^3\right)^2 = x^{3\times 2} = x^6$ and the rule is that *powers raised to powers* means we *multiply* the powers.

There are other patterns, sometimes also called rules or laws, for performing algebra with powers which you will also come across and use at different times. The two main ones are:

- $(xy)^n = x^n y^n$, for example $(2a)^3 = 8a^3$, and
- $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$, for example $\left(\frac{x}{7}\right)^2 = \frac{x^2}{49}$.

Finally these two facts you'll come across for powers are less rules and more mathematical notation choices:

For all x,

$$x^0 = 1,$$

except when $x = 0$, in which case 0^0 is mathematical nonsense.

For all x and all n,

$$x^{-n} = \frac{1}{x^n}.$$

This can be thought of as a definition of negative powers if you wish.

Example 1
Simplify

$$\left(xy^2\right)^3 \div y$$

Answer: First we need to expand the bracket (BODMAS), remembering that every term in the bracket needs to be cubed. So we get

$$(x)^3(y^2)^3 \div y$$

the next step is to use our third power law to rewrite $(y^2)^3$ as y^6. Then finally we use the second power law (about division) to replace $y^6 \div y$ with $y^{6-1} = y^5$ giving our final answer of

$$x^3y^5.$$

Note that the x and y terms are different letters and so are not combinable using the power laws.

Example 2
Simplify
$$\frac{3x^2y^3z^{-2}}{(xy^{-3}z)^2}$$
Answer: Again we begin by expanding the bracket on the bottom, we are using the fact that
$$(abc)^2 = a^2b^2c^2$$
to give us
$$\frac{3x^2y^3z^{-2}}{(x)^2(y^{-3})^2(z)^2}$$
On the bottom we now need to replace $(y^{-3})^2$ with y^{-6}, using our third law again. So we have reached:
$$\frac{3x^2y^3z^{-2}}{x^2y^{-6}z^2}$$
Now we can treat the x, y and z parts separately.
Our overall x terms looks like:
$$x^2 \div x^2 = x^{2-2} = x^0 = 1$$
Our overall y term looks like:
$$y^3 \div y^{-6} = y^{3--6} = y^{3+6} = y^9$$
note we were careful here to subtract (not add) -6, so two negative signs gave a $+$.
Finally our overall z term will be:
$$z^{-2} \div z^2 = z^{-2-2} = z^{-4}.$$
So overall (not forgetting the initial factor of 3) we obtain:
$$\frac{3x^2y^3z^{-2}}{(xy^{-3}z)^2} = 3y^9z^{-4} = \frac{3y^9}{z^4}$$
and the x term disappeared as it was identical on top and bottom (and so cancelled perfectly).

Practice
Simplify these 8 formulae and then put them into 4 matching pairs.

i. $x \times x \times y \times y \times y^2 \times \frac{1}{z^2}$
ii. $(xyz)^3x^{-1}z^{-5}$
iii. $xy^4z \div y^2$
iv. $x^3x^{-5}y^2y^{-3}z^{-2}z^1$
v. $x^2y^3 \div z^2$
vi. $\frac{x^{-2}y}{y^2z}$
vii. $\frac{(xyz)^2}{xz}$
viii. $(xy)^2\left(yz^{-1}\right)^2$

```
#> [Embedded question appears here in html version
#>  the hyperlink is provided below]
```

Link to Numbas on the web

3.1.4 Grouping 'like' terms

This is one of the key early skills required to perform algebra. We all know that for a sum like $3-4+8-16+7$, arithmetic can be used to simplify this into a single number (-2 in this example). This is because all the terms are just constants.

If, however, you are presented with the formula $3 - x + 8 + 5x - 7$ then the simplification will not be just a number. You need to identify terms which are 'like' (or 'alike') because they contain unknown variables in exactly the same format as each other. In this example the 3, 8 and -7 are alike as they are all just constants and don't mention x. Similarly the $-x$ and $5x$ are alike because they are both of the format 'constant$\times x$'. We can combine terms when they will look the same if you overlook different constants on the front. The constant on the front of an unknown variable is called the *coefficient*, e.g. the coefficient of $7x$ is 7 and the coefficient of $-4y$ is -4. Terms which have different powers of an unknown variable are regarded as in a different format.

More generally, all of the following are different formats to each other

$$x, x^2, x^3, x^4, x^{1.5}, x^{-7}, xy, xz, xy^2, ...$$

The list of different formats could go on forever, but the key idea is as follows.

Two terms are of the same format if when they are added together you get another object of exactly the same format. When dealing with variables and their powers, they are the same format if they have identical variables and powers.
So x^2 and y^2 are not alike, but $3xz^2$ and $-9xz^2$ are alike.

It's almost always a good idea, before diving into performing algebra and re-arranging an equation, to first group like terms together and combine their coefficients.

Some quick practice

- $5 + x - 8 + x = -3 + 2x$
- $x^2 - 4x + 3 + 7x = x^2 + 3x + 3$
- $8 - x^2 + 2\left(x^2 + 5\right) = x^2 + 18$
- $x(x - y) + y(z + x) = x^2 + yz$
- $7x^2 - 4xy^2 + x^2y + 3xy - 8y^2 + xz - xyz + 3y^2x =$
  $7x^2 - xy^2 + x^2y + 3xy - 8y^2 + xz - xyz$

In all these examples the second version is fully simplified, all presented terms are of different formats. You should check them carefully to see that you agree.

When progressing to more advanced algebra, with formulae that include functions like $\sin(x)$, and $\log(y)$ we will need to expand our definition of *like* terms. The good news is that once functions are involved, only different constants on the front of the function mean terms are alike, otherwise they are different. So $3\sin(4x) + 7\sin(4x)$ can be combined into $10\sin(4x)$ but $3\sin(x) + 4\sin(2x)$ cannot be combined.

3.2 Common algebra mistakes

It's always slightly dangerous to show incorrect algebra, as it might make some poor quality memories for you. However, some errors are so common when working with brackets, powers and fractions that it can still be worthwhile to show you them in action. Make sure to spend longer looking at the 'Correct algebra' side of this table. Perhaps even start with the right-hand-side and then look at the left to see how not to do it.

Common mistake (i.e wrong!)	Correct algebra
$(2x)^3 = 2x^3$	$(2x)^3 = 8x^3$
$(x+y)^2 = x^2 + y^2$	$(x+y)^2 = x^2 + y^2 + 2xy$
$\frac{a+b}{c+d} = \frac{a}{c} + \frac{b}{d}$	$\frac{a+b}{c+d} = \frac{a}{c+d} + \frac{b}{c+d}$
$a^{m+n} = a^m + a^n$	$a^{m+n} = a^m \times a^n$

Common mistake (i.e wrong!)	Correct algebra
$\frac{1}{2x^3} = 2x^{-3}$	$\frac{1}{2x^3} = \frac{1}{2} \times \frac{1}{x^3} = \frac{1}{2}x^{-3}$

3.3 Division of algebraic expressions

The main idea to remember here is that if you break up a fraction which contains multiple terms on the top (often called the numerator) separated by + and − signs, the bottom (also called the denominator) remains the same.

For example,

$$\begin{aligned}\frac{5x^3 - 3x^2 + 7}{x^2} &= \frac{5x^3}{x^2} - \frac{3x^2}{x^2} + \frac{7}{x^2}\\ &= 5x - 3 + \frac{7}{x^2}\end{aligned}$$

Or if we're trying to be a little cleverer, an example like this:

$$\begin{aligned}\frac{x+7}{x+2} &= \frac{x+2+5}{x+2}\\ &= \frac{x+2}{x+2} + \frac{5}{x+2}\\ &= 1 + \frac{5}{x+2}\end{aligned}$$

More complicated expressions like this second example, where the bottom (denominator) isn't just one term, can also be tackled via what is called 'long division'.

This is normally required if the denominator contains at least one + or − sign. Such as trying to simplify

$$\frac{x^3 + 7x + 3}{x+2}$$

Trying to divide $x + 2$ into $x^3 + 7x + 3$ can be done, with a remainder but it's a fairly tedious procedure that can be looked up if required. The main method used for this is called *synthetic division*, it is an efficient way of performing divisions without too many algebraic letters, especially then the starting coefficients on the higher powers are both 1. However, we won't go into details here.

The final answer here [5] is:

$$\frac{x^3 + 7x + 3}{x+2} = x^2 - 2x + 11 - \frac{19}{x+2}$$

3.4 Exponential Functions and Logarithms

The topics of exponentials and logarithms go together because they are essentially each others' reverse procedure. We shall look at them separately but then exploit this idea to solve equations.

[5] which you would need to go and look up the procedure to calculate

3.4.1 Exponential functions

As mentioned earlier exponentiation, powers, indices and orders are all terms referring to the same mathematical procedure. This procedure can be used to create a number of interesting functions and graphs that appear in many real world applications of mathematics. The grand family of such functions are **exponential functions**.

Exponential functions all have this general structure:

$$y = b^x$$

where b is some chosen constant. The variable x is being considered to vary, causing changes in y.

It is only worth studying such functions when $b > 0$ and $b \neq 1$ (Can you see why $b = 1$ is very boring?). We use the letter b in our example because the number at the bottom of the exponential structure (i.e. the bottom of the little tower of symbols) is called the **base** of the exponential function. This same term will be used later when discussing logarithms (which will be the reverse of exponentiation).

The most commonly studied members of the exponential family are those with base 10 and base e [6], though in Computing base 2 is also very popular.

The Exponential Function
The function with base $= e$, as defined above, is called **the** exponential function because it is so important in mathematics. The function is written

$$y = e^x,$$

and it has the amazing property that the gradient of its graph is always equal to the graph's height.

When you later study differentiation you will see this fact again, i.e. that the derivative of e^x is just itself!

3.4.2 Logarithms

One piece of good news it that logarithm is usually abbreviated to log to save remembering how to spell it! The logarithm of a number is just another number, normally a never-ending decimal. Much like other functions you will have come across like sine, cosine, tangent or squaring, the logarithm is just another function which takes inputs and returns one output per input.

We shall start with assuming base $b = 10$ to save any initial complications. Below are some sample values from the log function.

Values of log (in base 10) showing first five decimal places only
$\log(0.2) = -0.69897...$
$\log(0.5) = -0.30103...$
$\log(1) = 0$
$\log(2) = 0.30103...$
$\log(3) = 0.47712...$
$\log(4) = 0.60206...$
$\log(5) = 0.69897...$
$\log(6) = 0.77815...$
$\log(7) = 0.84509...$

[6] a constant equal to 2.71828...

Values of log (in base 10) showing first five decimal places only
$\log(8) = 0.90309...$
$\log(9) = 0.95424...$
$\log(10) = 1$

Only $\log(1) = 0$ and $\log(10) = 1$ give nice values, the rest appear to be a mess. However, certain patterns can be spotted which are special cases of general rules you need to learn to be able to use logs well in algebra. Before moving on see if you can spot any relationships between the log values above.

The formal definition of what the logarithm function calculates is as follows:

Definition of the logarithm

$$\log_b(x) = \textit{The power you need to raise } b \textit{ to, to get } x$$

i.e. if $\log_b(x) = 4$ then

$$b^4 = x$$

Note that the base in use is written as a subscript after the word log, as in $\log_b$.

This definition generally isn't very useful, because it's only useful for certain special values. Sure it's easy to now say that $\log_{10}(1000) = 3$ because we know that $10^3 = 1000$ but no-one really knows what power of 10 equals 75, it's just some disgusting decimal[7].

Far more useful is to understand that logarithms (using base 10) perform the opposite procedure to *raising 10 to the power of something*.

If we begin with $z = 3.4$ and we calculate $10^z = 10^{3.4} = 2511.89...$ then we now know that $\log(2511.89) = 3.4$. Taking the logarithm of a number means we are determining what power of the base we need to get that number.

We have assumed until now that this base is 10, but in fact any positive number[8] can be used for the base b. This is exactly like our study of the family of exponential functions above, where we also used b as the base, and we considered other values in addition to just $b = 10$. Again the standard bases used are $b = 10$ and $b = e = 2.71828...$. The good news is that it is very rare you would ever want to use a mixture of bases in the same question/problem, so you normally decide up front which base to work in and just write log. However, technically you will find authors write:

$$\log_b(x)$$

as the standard notation to signify that a base of b is being used. Perhaps most annoying of all, when first studying logarithms, is that when using base $b = e$ authors often use ln() instead of $log_e()$ because log with a base of e has historically been called the *natural logarithm* and [9] the first authors called it 'logarithmus naturalis' hence the reversal of the letters N and L.

You will find a log button on all scientific calculators, it normally shares a button with the reverse exponentiation procedure. So here are a few questions for you to practice using them both,

[7] my calculator says it's 1.875061...
[8] except $b = 1$, can you see why?
[9] legend has it

Practice
Calculate the following on your calculator, giving your answer correct to 4 decimal places (remember to round up if necessary)

(1) $\log_{10}(34)$
(2) $\log_{10}(42)$
(3) $\log_{10}(17)$
(4) $\ln(5)$
(5) $\ln(2)$

To check your answers, see the next exercise.

Practice
Calculate the following on your calculator, giving your answer correct to 1 decimal place (remember to round up if necessary)

(1) $10^{1.2304}$
(2) $10^{1.5315}$
(3) $10^{1.6232}$
(4) $e^{0.6931}$
(5) $e^{1.6094}$

Now go back and match up these answers with your answers to the previous exercise. Note that they won't match perfectly as the numbers were rounded to 4 decimal places, but they will be very close.

3.4.3 Log laws

The laws for powers, which we met earlier in equations (3.2), (3.3) and (3.4) have completely equivalent versions for logarithms, though they are seen as more difficult to memorize than for powers, since it's less easy to visualize their meanings. In all the laws the assumption is that all logs present have the same base, so we shall omit it for clarity.

Law 1: For all a, b, both positive...

$$\log(a \times b) = \log(a) + \log(b) \tag{3.8}$$

```
#> [Video appears here in html version
#>  the hyperlink is provided below]
```

Alternative direct video link:

https://gcu.planetestream.com/View.aspx?id=3305~4l~LrqjFeRf

For comparison with the power laws, this is the log version of the $x^a \times x^b = x^{a+b}$ power law, where multiplication becomes addition.

The second log law (much like the second power law) can be worked out using the first and third log laws, but it can also be learned too.

Law 2: For all a, b, both positive...

$$\log\left(\frac{a}{b}\right) = \log(a) - \log(b) \tag{3.9}$$

```
#> [Video appears here in html version
#>  the hyperlink is provided below]
```

Alternative direct video link:

https://gcu.planetestream.com/View.aspx?id=5990~4x~8AS88NtP

This second law is the log version of the $x^a \div x^b = x^{a-b}$ law, where division becomes subtraction.

The third log law concerns the log of a power tower.

Law 3: For all positive x, and all n positive or negative...

$$\log\left(x^n\right) = n\log(x) \tag{3.10}$$

```
#> [Video appears here in html version
#>  the hyperlink is provided below]
```

Alternative direct video link:

https://gcu.planetestream.com/View.aspx?id=3306~4m~m2FUpdbq

This law can also be compared directly with the third power law, although the similarity is less obvious. The parallel with $(x^a)^b = x^{ab}$ lies in the fact that a power has become a product.

All three of these laws can prove very useful when manipulating equations with algebra involving logs. The most useful of the three laws is generally the third one, although the first law gets frequent use too (see Section 3.4.4).

3.4.4 Solving equations using logs and powers

Most real-world applications of logs and exponentials comes from the real-world providing an example of what is called a 'power law' relationship. The good news is that power law relationships are really just examples of exponential functions. The mathematical problem is then often to solve an equation for an unknown variable which appears 'in the power' (i.e. at the top of the tower). For example,

$$y = 4e^{-3x} \tag{3.11}$$

where we know that $y = 15$ and we need to find x. Or more generally, we might just want to re-arrange to find x in terms of y.

In all such examples the main idea is to exploit the fact that logarithms and powers are opposite procedures *as long as the same base is used*.

Taking logs
Starting with an equation and then equating the log of the left-hand-side with the log of the right-hand-side is called **taking logs** and is the standard approach for simplifying powers in algebra.
Specifically if we start with

$$A = B$$

then by *taking logs* we reach:

$$\log(A) = \log(B) \tag{3.12}$$

where A and B can be complicated expressions.

In our example (3.11) we can take logs (using base e because that is the base already present on the right-hand-side) to turn $y = 4e^{-x}$ into

$$\log_e(y) = \log_e(4e^{-3x})$$

it would be standard to have written *ln* (for the natural logarithm) here, rather than the long-version $\log_e$.

On the right-hand-side we can now use the first log law, and then the third law and this equation becomes

$$
\begin{aligned}
\log_e(y) &= \log_e(4) + \log_e(e^{-3x}) \\
&= \log_e(4) - 3x\log_e(e) \\
&= \log_e(4) - 3x
\end{aligned}
$$

using the fact that $\log_e(e) = 1$[10]. In fact, this is why we chose the base of our log to match the existing base If we wish, it wouldn't be difficult to re-arrange this equation to make x the subject.

It's always true that

$$\log_e(e^z) = z, \text{ for any value of } z.$$

So you can almost bypass the third log law if your bases match, and just know that log will just destroy the tower, and leave only the power.

Here's a demonstration of the procedure with commentary:

```
#> [Video appears here in html version
#>  the hyperlink is provided below]
```

Alternative direct video link:

https://gcu.planetestream.com/View.aspx?id=3309~4p~QdzxumJx

Practice
Use the three Log Laws to break up / expand each of these five logarithms, and then match them up with the offered answers below.

(1) $\log\left(x^3y^{-2}z\right)$
(2) $\log\left(x^2y^3z^4\right)$
(3) $\log\left(x^{-1}y^2z^3\right)$
(4) $\log\left(\frac{x^2z^3}{y}\right)$
(5) $\log\left(\frac{x^4z}{y^3}\right)$

Answers: (not in the same order)

(a) $-\log(x) + 2\log(y) + 3\log(z)$
(b) $4\log(x) - 3\log(y) + \log(z)$
(c) $2\log(x) + 3\log(y) + 4\log(z)$
(d) $3\log(x) - 2\log(y) + \log(z)$
(e) $2\log(x) - \log(y) + 3\log(z)$

Practice
Now try the reverse process, for each of these formulae try and join the elements together into a single logarithm (again using the three Log Laws). Then match them to the answers below:

(a) $\log(x) + \log(y) - \log(z)$
(b) $3\log(x) - 2\log(y) + \log(z)$
(c) $2\ln(x) + 3\ln(y) + 4\ln(z)$
(d) $-\ln(x) - 2\ln(y) + \ln(z)$

Answers: (not necessarily in the same order)

(1) $\log\left(\frac{xy}{z}\right)$ or $\log(xyz^{-1})$
(2) $\log\left(\frac{x^3z}{y^2}\right)$ or $\log(x^3y^{-2}z)$
(3) $\ln\left(\frac{z}{xy^2}\right)$ or $\ln(x^{-1}y^{-2}z)$
(4) $\ln\left(x^2y^3z^4\right)$

[10] What power of e equals e? One!

```
#> [Embedded question appears here in html version
#>  the hyperlink is provided below]
```

Link to Numbas on the web

3.5 Operations and their inverses

We have already talked about addition and subtraction as opposite procedures, and the same for multiplication and division. Formally in mathematics we call reverse procedures **inverses**. An operation and its inverse, when applied in sequence, effectively cancel each other out as if nothing ever happened. For example, if we start with x and multiply by 7, then divide by 7, we get back to x.

In fact it doesn't even matter which way around we perform these two steps, dividing by 7, then multiplying by 7 also leaves us back where we started. Here is a table of common inverses, which you can and will use when performing algebra. Columns 1 and 2 can be swapped if desired.

First operation	Second operation	Demonstration
$+a$	$-a$	$x+4-4=x$
$\times b$	$\div b$	$\frac{x\times b}{b}=x$
Squaring	Square-rooting[11]	$\sqrt{x^2}=x$
Cubing	Cube-rooting[12]	$\sqrt[3]{x^3}=x$
$10^{\square}$	$\log_{10}(\square)$	$10^{\log_{10}(x)}=x$ or $\log_{10}(10^x)=x$
$\sin(\square)$	$\sin^{-1}(\square)$	$\sin^{-1}(\sin(x))=x$
$\cos(\square)$	$\cos^{-1}(\square)$	$\cos^{-1}(\cos(x))=x$

3.6 Manipulating formulae and solving equations

This section contains the main skill development topic in the chapter of Algebra. It concerns decision-making, and accurate performance of algebraic manipulations of equations to try and get 'answers'. Sometimes it may also not always be obvious what the answer looks like when you start, so this can be part of the challenge too.

The phrase, to 'solve an equation' can have a variety of target outcomes. When asked to 'solve an equation for x', which is also described as re-arranging the equation in algebra to make x *the subject of the equation*, we typically get two cases:

- Case 1: There are no other symbols in the formula so we can find exactly the value(s) of x.
- Case 2: There are other unknown letters in our formula, in which case our answer will be $x =$ some algebra (not mentioning x).

Becoming skilled at algebra takes practice. When just reading notes and model solutions to questions you don't develop the practical memories and patterns of personally performing the algebra. So **repeated and continuous practice is vital to developing your skills**.

[11] raising to the power 1/2
[12] raising to the power 1/3

3.6.1 Fundamental idea: Keeping an equation balanced

The first key rule whenever performing algebra is to always **do the same thing to both sides of an equation**. For example, we saw this principle in action earlier when **taking logs**, in (3.12).

The definition of *equation*, is a formula on the left-hand-side equalling a formula on the right-hand-side. So if you were to add 9 to both sides then they will remain equal to each other, as they will both have just increased by 9.

Here are a sample of typical procedures that keep an equation balanced:

1. Add the same thing to both sides;
2. Subtract the same thing from both sides;
3. Divide both sides by the same thing;
4. Multiply both sides by the same thing;
5. Square both sides;
6. Square root both sides[13];
7. Take inverse sine (often written $\sin^{-1}$) of both sides; or $\cos^{-1}$ or $\tan^{-1}$;
8. Take the log on both sides (with the same base);
9. Exponentiate both sides (with the same base).

There are more operations it's possible to do, but this list covers the main ones you'll use. Indeed the first four, using $+, -, \div$ and $\times$ will make up the vast majority of algebraic steps you ever perform, so you should get good and confident with performing them. However, skill with all the procedures in this list is required to become strong at algebra.

I suggest you watch these four short videos in order to see the first four operations in action. Notice that in the simple examples chosen we generally use more than one of these operations to get our answer for x. Generally when dividing or multiplying both sides of an equation by something we later need to also add or subtract from both sides to clean up our answer.

Adding the same quantity to both sides of an equation

```
#> [Video appears here in html version
#>  the hyperlink is provided below]
```

In this example, the clever choice is to add 13 to both sides because this turns $x - 13$ into just x, and we so get an equation that says $x =$ something (which is our target).

Subtracting the same quantity from both sides of an equation

```
#> [Video appears here in html version
#>  the hyperlink is provided below]
```

In this example, the clever choice is to subtract 12 to both sides because this turns $x + 12$ into just x, and we so get an equation that says $x =$ something (which is our target).

Dividing both sides of an equation by the same quantity

```
#> [Video appears here in html version
#>  the hyperlink is provided below]
```

In this example, there were two clever choices made. First we subtract 5 from both sides to get $3x =$ something. Then we choose to divide both sides by 3 which turns the $3x$ on the left into just x. The division by 3 cancels with the multiplication by 3 visible in $3 \times x$.

Multiplying both sides of an equation by the same quantity

```
#> [Video appears here in html version
#>  the hyperlink is provided below]
```

[13] Warning: see later

The clever idea here is to choose to multiply both sides by $x + 1$. The objective is to get our target x out from the bottom of the fraction, and multiplying a fraction by its bottom always simplifies a fraction. In this case $\frac{9}{x+1} \times (x + 1) = 9$ in a special example of the general rule that $\frac{A}{B} \times B = A$ for any A, B.

Here are direct links to the four videos:

- Direct Link to Adding Video
- Direct Link to Subtracting Video
- Direct Link to Dividing Video
- Direct Link to Multiplying Video

In all of these examples above we had only one unknown variable x, there is nothing different in procedure if other letters are present, they are treated as algebra variables like any other elements.

```
#> [Embedded question appears here in html version
#>  the hyperlink is provided below]
```

Link to Numbas on the web

Warning 1 When dividing both sides of an equation by something it's important to note that you're never allowed to divide by zero. This scenario is easy to avoid if you are dividing by known values, but care is needed if the term you divide by contains any unknowns. If this situation does occur you should consider the case where the term is zero separately.
For example, to make x the subject of this equation:

$$y = x(1 - z) \tag{3.13}$$

we want to divide by $1 - z$, which is fine unless $1 - z = 0$. So as long as $1 - z \neq 0$

$$x = \frac{y}{1 - z}.$$

However, if we are in the case where $1 - z = 0$ then we must not try and divide by $1 - z$, instead we look back at (3.13) which now just says $y = 0$ and we cannot learn the value of x from the equation.

Warning 2 When square-rooting both sides of an equation it's important to remember that there is always a positive and negative square root of a number.
For example, $x^2 = 16$ has two solutions, $x = 4$ and $x = -4$. The standard way to write this is $x = \pm 4$.[a]

[a]pronounced plus-or-minus 4

3.6.2 Some detailed worked examples:

Example 1 Given the equation $34 = 2\,(4 - 5x)$ we can perform algebra (typically using inverses, see 3.5) to solve for x: (the steps taken on both sides are written in the margin)

$$\begin{aligned} 34 &= 2\,(4 - 5x) && \\ 17 &= 4 - 5x && (\div 2) \\ 13 &= -5x && (-4) \\ -\frac{13}{5} &= x && (\div -5) \end{aligned}$$

and we have solved to find x exactly, it's $x = -\frac{13}{5} = -2.6$.

Example 2 The standard pendulum equation relating the *period* (T), the pendulum *length* (L) and gravitational constant g is normally written

$$T = 2\pi\sqrt{\frac{L}{g}}.$$

Re-arrange this equation to find the pendulum length as a function of the period, i.e. make L the subject.
We need to remove the $\sqrt{}$ symbol around L. We could choose to divide through by 2π before squaring both sides, but we can also just dive straight in.

$$\begin{aligned}
T &= 2\pi\sqrt{\frac{L}{g}}, \\
T^2 &= \left(2\pi\sqrt{\frac{L}{g}}\right)^2, && \text{(square both sides)} \\
T^2 &= 4\pi^2\frac{L}{g}, && \text{(expand bracket)} \\
\frac{T^2}{4\pi^2} &= \frac{L}{g}, && (\div 4\pi^2) \\
\frac{gT^2}{4\pi^2} &= L. && (\times g)
\end{aligned}$$

And we are finished.
Extension: The period T is actually the reciprocal of *frequency*, f, i.e. $T = 1/f$. Express L in terms of f.
This is straightforward as we can just replace T with $\frac{1}{f}$ and we don't need to restart. Noting that $T^2 = \frac{1}{f^2}$ we get

$$L = \frac{g}{4\pi^2 f^2}.$$

Example 3 In optics the Lens equation relating *object distance* (a), *image distance* (b) and *focal length* (f) is written

$$\frac{1}{f} = \frac{1}{a} + \frac{1}{b}.$$

In fact this same equation also occurs in electronics, for resistors in parallel and for capacitors in series.
Re-arrange this equation to make b the subject.
There are multiple ways to approach this, we shall look at just one way. We shall begin by moving the terms around (using subtraction) to get one side containing only b.

$$\begin{aligned}
\frac{1}{f} &= \frac{1}{a} + \frac{1}{b}, \\
\frac{1}{f} - \frac{1}{a} &= \frac{1}{b}, && \text{(subtracting } \tfrac{1}{a}\text{)}
\end{aligned}$$

Quick warning: It might be tempting to try and 'flip'[a] everything here, i.e. turn all fractions upside-down. Such a procedure is permitted if each side is just a single fraction, but that's not true of the left here. The 'flip' of the right is indeed b but the 'flip' of the left is $\frac{1}{\frac{1}{f}-\frac{1}{a}}$[b]
One way to proceed if our equation said $A = \frac{1}{b}$ would be to multiply both sides by b (which would have the effect of moving the b to the left hand side), and then dividing both sides by A (which would

have the effect of moving the A to the right hand side, on the bottom of a fraction). However, our object on the left isn't a simple single object A but a difference of two fractions. So we next choose to combine the left two terms together into one fraction.
For this we need a common bottom (denominator). We can always use the product of the two denominators as our new denominator, so we choose this, i.e. $f \times a$. We multiply the top and bottom of each fraction to leave them unchanged in value, but both have denominator of $f \times a$:

$$\frac{1}{f} - \frac{1}{a} = \frac{a}{fa} - \frac{f}{fa} = \frac{a-f}{fa}.$$

Returning to our original equation we therefore have:

$$\frac{a-f}{fa} = \frac{1}{b} \tag{3.14}$$

Now we are ready to proceed with our plan, with $A = \frac{a-f}{fa}$.

$$\begin{aligned} \frac{a-f}{fa} &= \frac{1}{b}, \\ b \times \frac{a-f}{fa} &= 1, && \text{(multiplying by } b\text{)} \\ b &= 1 \div \frac{a-f}{fa}, && \text{(dividing by } A\text{)} \\ b &= 1 \times \frac{fa}{a-f}, && \text{(standard fraction algebra)} \\ b &= \frac{fa}{a-f} \end{aligned}$$

You'll see how we only needed multiplication and division, after our initial subtraction. However, we needed some good fraction skills along the way too.
It was possible to shortcut the final steps after Equation (3.14) and instead 'flip' both sides of the equation, jumping straight to the final line. This is permitted when each side contains a single fraction as it did here, but for illustrative purposes we have taken the long route this time.

[a] formally called taking the reciprocal
[b] and this is *not* equal to $f - a$!

Example 4 The decay of the radioactive element radium is modelled by the formula

$$A = A_0 (0.5)^{\frac{t}{1620}}$$

where A_0 is the initial amount of radium and A is the amount present after t years.
(i) How much radium remains in a 1kg sample after 1000 years?
(ii) How long would it take for a 1kg sample to decay to 0.01kg?
Answers:
(i) Set $A_0 = 1$, and $t = 1000$ in the given formula and evaluate (with calculator):

$$A = 1 \times (0.5)^{\frac{1000}{1620}} = 0.6519\text{kg}$$

(ii) Set $A_0 = 1$, $A = 0.01$ in the given formula and solve for t:

$$\begin{aligned} 0.01 &= (0.5)^{\frac{t}{1620}} \\ \ln(0.01) &= \ln\left((0.5)^{\frac{t}{1620}}\right) && \text{(taking logs)} \\ \ln(0.01) &= \frac{t}{1620}\ln(0.5) && \text{(using log law 3)} \end{aligned}$$

So, (re-arranging for t using multiplication and division, then our calculator)

$$t = \frac{1620 \ln(0.01)}{\ln(0.5)} = 10763 \text{ years.}$$

Example 5 The current, i, in the branch of an electronic circuit changes with time, t, in line with the following formula:

$$i = i_0 e^{-kt}.$$

The initial current (i.e. when $t = 0$) is 15mA, and it takes 4.7s for the current to drop to 7.5mA (i.e. half it's initial value).

(i) From the information given, determine the values of the parameters i_0 and k.
(ii) Determine the current when $t = 6.5$s.
(iii) Determine t when the current is 25% of its initial value.

Answers:

(i) We put $t = 0$ and $i = 15$ into our formula to see what we can learn:

$$15 = i_0 e^{-k\times 0}.$$

But $e^{-k\times 0} = e^0 = 1$, so

$$15 = i_0 \times 1,$$

so we know that $i_0 = 15$, and the formula actually is known to look like $i = 15e^{-kt}$. We also know that when $t = 4.7$ then $i = 7.5$, we can input these values next to find k:

$$\begin{aligned} 7.5 &= 15e^{-4.7k} && \\ 0.5 &= e^{-4.7k} && (\div 15) \\ \ln(0.5) &= \ln(e^{-4.7k}) && \text{(taking logs)} \\ \ln(0.5) &= -4.7k && (\ln \text{ and } e^{\square} \text{ are opposites}) \end{aligned}$$

So,

$$k = \frac{\ln(0.5)}{-4.7} = 0.147478123.$$

So our completed formula is

$$i = 15e^{-0.14748t}.$$

(ii) Now we have our formula, this part is easy. We are being asked for i when $t = 6.5$ so we just calculate

$$i = 15e^{-0.14748\times 6.5} = 5.75 \text{ milliamps}$$

(iii) 25% of 15 is 3.75 so we are looking for the value of t which makes $i = 3.75$, i.e. to solve

$$3.75 = 15e^{-0.14748t}$$

The steps are similar to those above,

$$\begin{aligned} 0.25 &= e^{-0.14748t} && (\div 15) \\ \ln(0.25) &= -0.14748t && \text{(taking logs)} \\ \frac{\ln(0.25)}{-0.14748} &= t && (\div -0.14748) \end{aligned}$$

So $t = \frac{\ln(0.25)}{-0.14748} = 9.40$ seconds is the time until current is 25% of original current.

3.7 Linear equations: additional tips

The equation of a line looks like $y = a + bx$ or $y = mx + c$, or indeed anything of the following format:

$$\textit{unknown} = x \times \text{constant} + \text{constant} \tag{3.15}$$

The *unknown* variable can be given any name, but often we use y, so we get examples like $y = 2x + 1$, or

$y = -6x + 5$ or even $y = 0.243x - 0.946$. Similarly if the unknown variable which varies isn't called x that's fine too.

The key fact is that the variable, x, appears by itself (perhaps with a constant factor on the front) and not inside another function like $\sin(x)$ or $\log(x)$, and it is not raised to any powers like x^2, x^3 or even $x^{0.2}$. Here's an informal reminder of what a linear equation is.

If the variable x appears only as "constant$\times x$" , not to any powers, and not inside any functions, then we say a formula is **linear in** x.
A **linear equation** is one where the variables concerned only appear in linear fashion (as above).

Linear equations are some of the easiest equations to re-arrange, and thus solve, for the linear variable, because you can always use addition and subtraction to move all the terms of the target variable onto one side. Then with an appropriate division of both sides you can obtain the variable by itself. This approach even works if the linear variable appears in a few different terms, but it will require some factorizing before the division.

Example

$$4y - 3x + 9 = 5x + x\log(7)$$

with addition and factorization becomes

$$4y + 9 = x\,(8 + \log(7))$$

division then results in our answer:

$$\frac{4y + 9}{8 + \log(7)} = x$$

3.8 Quadratic equations

Re-arranging quadratic equations can be done much like any other equations. However because x^2 terms and x terms cannot be combined (they are not *like terms*) you won't end up with an equation that says $x =$ something. Instead you'll end up with an equation like what is called the *standard format* for a quadratic equation:

$$Ax^2 + Bx + C = 0 \tag{3.16}$$

Here A, B and C are constants. For example, beginning with $2x^2 - 8x = x^2 - 7$ we can re-arrange to reach $x^2 - 8x + 7 = 0$, which is in standard format with $A = 1$, $B = -8$ and $C = 7$. Note also that we have moved the constant to the left-hand-side too to obtain $= 0$ on the right.

The ability to match up the A, B and C constant values with your specific example is important for using the quadratic formula (see Section 3.10).

Although the standard format above is fully simplified, it is not the answer to the question 'solve for x'. To find the values of x which make the equation true we need to use other techniques. There are three very popular approaches to solving quadratic equations:

- Factorization
- Completing the square
- The Quadratic formula

First, a quick summary of the advantages and disadvantages of the different methods available.

Factorization:
Advantages: Very quick, easy to read-off answers
Disadvantages: Not always possible[a], often requires some searching

[a]without first solving the problem another way!

Completing the square:
Advantages: Very useful for plotting/graphing, Always works
Disadvantages: Multiple algebraic steps, messy when $A \neq 1$

The Quadratic Formula:
Advantages: Always works, just one formula to learn, fast (with practice) Disadvantages: One messy formula to learn, provides no real insights

3.8.1 Factorization of a quadratic

This is the process of re-writing a quadratic formula as a product of two brackets each containing an x term. Generally it looks like this: (or nearby variations)

$$Ax^2 + Bx + C = A\left(x + P\right)\left(x + Q\right)$$

where all the capital letters represent constants.

We'll start by just looking at the easier (but still frequent) case where $A = 1$ above. Here's an explicit example:

$$x^2 - 5x + 6 = (x - 2)(x - 3)$$

The great news is that in this factorized format it becomes easy to find the solutions, for x, in the equation equal to zero, i.e.

$$x^2 - 5x + 6 = 0 \qquad \text{becomes} \tag{3.17}$$

$$(x - 2)(x - 3) = 0 \tag{3.18}$$

The only solutions to an equation that says

$$(\text{Bracket 1}) \times (\text{Bracket 2}) = 0$$

occur when (Bracket 1) $= 0$ or when (Bracket 2) $= 0$. Since **the product of two numbers can only be zero if (at least) one of them is zero**.

This makes it easy to find the solutions for x by just looking to see what values of x make each bracket equal to zero. In (3.18) we had $(x - 2)(x - 3) = 0$ so the two solutions are when:

- $(x - 2) = 0$, which means $x = 2$, and
- $(x - 3) = 0$, which means $x = 3$.

Returning to the specific factorization procedure itself, here are a few more examples to help us spot the pattern required for factorizing.

$$
\begin{aligned}
x^2 + 2x - 15 &= (x-3)(x+5) \\
x^2 + 4x + 3 &= (x+1)(x+3) \\
x^2 - 16 &= (x-4)(x+4) \\
x^2 - 13 + 42 &= (x-6)(x-7) \\
x^2 + 4x + 4 &= (x+2)(x+2)
\end{aligned}
$$

It is worthwhile to take a closer look at these factorizations to identify how, in each case, the numbers that appear on the left are related to the corresponding formula on the right and where the numbers actually come from. For example, in $x^2 + 2x - 15 = (x-3)(x+5)$ the $+2x$ comes from $+5x - 3x$ i.e. from $+5 - 3$ the sum of the numbers on the right hand side.

Notice that in all these examples (with $1x^2$ on the front) that the x coefficient is the **sum of the two numbers in the brackets**.
Notice also that the constant term is always **the product of the two numbers in the brackets**

You should expand out all of these examples by hand so that these two facts become obvious to you. The standard method for factorizing involves trying all sensible combinations of two numbers in the brackets to try and match the two rules above.

In the case that A doesn't equal 1, like in $2x^2 - 5x - 3$ it is more difficult to find the factorization, this is because you first have to choose how to get Ax^2 and then the rule involving the *sum matching the x-term doesn't work*. You can still do it by trial-and-error (as above), but it's harder.

For example, you start by writing

$$2x^2 - 5x - 3 = (2x + P)(x + Q)$$

and then try and find P and Q which work to give the left-hand side when expanded. Noticing that we need $P \times Q = -3$ narrows down the search to pairs of numbers which multiply to give -3, like $\{1, -3\}$ or $\{-1, 3\}$, and then trying the various combinations.

While in this case it does work to choose $P = 1$, $Q = -3$ there is a serious warning…

Factorization doesn't always work so easily. In fact **most quadratics cannot be easily factorized** by this method, so to find x you need to use a method that always works like "Completing the square" (Section 3.9) or the "Quadratic formula" (Section 3.10).

```
#> [Embedded question appears here in html version
#>  the hyperlink is provided below]
```

Link to Numbas on the web

And now for going beyond the factorized format to actually solving the quadratics:

```
#> [Embedded question appears here in html version
#>  the hyperlink is provided below]
```

Link to Numbas on the web

3.9 Completing the square

This method for solving quadratic equations always works, but is often not taught, with the quadratic formula just presented instead. You can survive solving all quadratics using just one of these two methods, although completing the square does provide extra insights into graph sketching.

We shall illustrate the technique through examples which should make the method pretty clear. Before we start, we should see what it means to say we have completed the square. Once you have completed the square for a quadratic you have a formula that looks like this:

$$\text{some quadratic formula} = (x+a)^2 + b$$

for some values of a and b which you will find precisely.[14]

Our first target is always to start with something in the standard $Ax^2 + Bx + C$ format and re-write it as something completely equivalent but in the completed square format. This is stage one. To solve a quadratic equation involves then performing some standard algebra to this completed square format to find the solution values of the unknown variable (we shall use x as is typical).

We can learn the two stages separately, first the process of re-writing a quadratic formula in completed square format. Then later we shall solve a quadratic equation using this format.

3.9.1 Completing the square: learning by example

Question: Complete the square for this quadratic formula: $x^2 - 6x - 7$

Answer: We begin by first checking that the coefficient of the x^2 term is 1[15]. If it says $2x^2$ or $-5x^2$ etc.. then first divide or take that factor out of an entire bracket (see later examples).

Now we identify what the coefficient is on the x term:

$$x^2 - 6x - 7$$

here it's -6. Take *half* of that value (it might be negative)[16] and create a squared-bracket which contains x followed by this number we just found (keep the positive or negative sign) as follows.

$$(x-3)^2$$

You will find that if you expand out this bracket you **almost** get the same as your starting quadratic formula. The only part that will be wrong[17] is the constant on the end.

Let's see it on our case above: the target was $x^2 - 6x - 7$ and if we expand our bracket we get

$$\begin{aligned}(x-3)^2 &= (x-3)(x-3) \\ &= x^2 + (-3)x + (-3)x + (-3)^2 \\ &= x^2 - 6x + 9\end{aligned}$$

So it's not quite true that $x^2 - 6x - 7$ equals $(x-3)^2$ but it's close. We then just need to find the difference between the two constants[18] and add or subtract the right value to make them match.

In our case we get

$$x^2 - 6x - 7 = (x-3)^2 - 16 \tag{3.19}$$

it was -16 because to change $+9$ into -7 we need to subtract 16.

This (3.19) is the final completed square format.

This format is fantastic for all kinds of mathematical uses. For example,

- The graph of $(x-3)^2 - 16$ is a standard $y = x^2$ graph shifted *right* by 3 and *down* by 16
- Because $(x-3)^2$ is never negative we know the lowest value of $x^2 - 6x - 7$ is $0^2 - 16 = -16$ *and* this minimum value occurs when the bracket is zero, i.e. when $x = 3$

[14] the signs could be negative rather than positive if a or b were negative
[15] Remember we normally write $1x^2$ as just x^2
[16] in this case we take half of -6 and get -3
[17] If you're really lucky this part will match and you are finished already.
[18] -7 and +9 in this case

In the graph presented below you can see the plot of $y = (x-3)^2 - 16$. It is literally just the standard base graph of $y = x^2$ shifted right by 3 and down by 16.

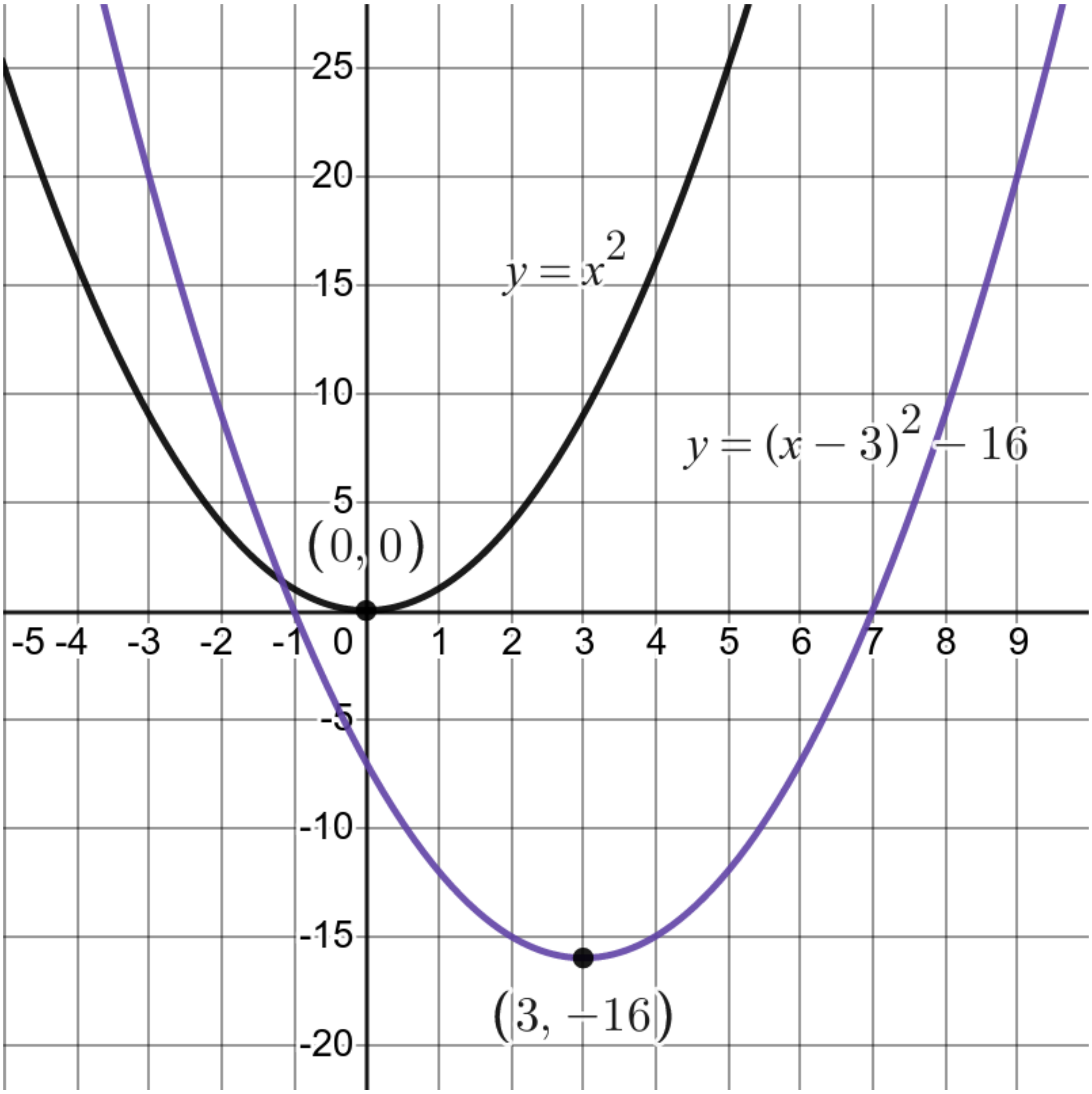

Figure 3.1: See the trough of the graph has been shifted right, and down

As seen in the graph above, the Completed Square format provided an easy way to find the minimum of a quadratic graph. It is in fact always true that the Completed Square format provides an easy way to find the minimum (or maximum[a]) of the graph.
This is often extremely useful information when studying a real-world problem containing a quadratic formula.
Suppose we want to study the graph of $y = Ax^2 + Bx + C$. If we begin by completing the square and are able to write it in the format:

$$A\left(x+M\right)^2 + N$$

where M and N are numbers we have calculated then...
the minimum (or maximum) of the graph will occur at $x = -M$ and the value when $x = -M$ will be $y = N$.
In words, the minimum (or maximum) of the graph always occurs at the value of x which makes the bracket equal to zero.
Hence,

- horizontally, the graph's trough (or peak) occurs when $x + M = 0$, i.e. $x = -M$,
- vertically, the height of the graph at its trough (or peak) is $0^2 + N$, i.e. $y = N$.

[a]if the graph is ∩-shaped

However, our real aim was to solve a quadratic equation[19], i.e. $x^2 - 6x - 7 = 0$ for this we still need to do a few short steps of algebra still.

$$\begin{aligned} x^2 - 6x - 7 &= 0 && \text{became...} \\ (x-3)^2 - 16 &= 0 && \text{which becomes...} \\ (x-3)^2 &= 16 \end{aligned}$$

So our first step is to move the constant to the other side so that it says *bracket squared* equals constant.

Now we square root both sides... remembering that there are two square roots of every positive number, in this case $+4$ and -4 both square into 16.

So we get

$$\begin{aligned} (x-3) &= +4 \text{ or } -4, && \text{so...} \\ x &= +7 \text{ or } -1 \end{aligned}$$

These general steps of algebra used here are the same every time you use completing the square.

Standard steps for the final equation solving part of completing the square:

- Move the constant to the right (i.e. add/subtract from both sides).
- Square-root both sides[a] (remembering there are two answers).
- Move the constant next to x to the right (i.e. add/subtract from both sides).
- You should be left with x as the subject.

[a]see Solubility Section 3.11 for when this is impossible

In comparison to factorization, which only worked some of the time, this method works all the time. The times that factorization works are when the answers are nice and neat (generally integer) answers for x. In general, however, you should expect your answers to a quadratic to contain surds (square root symbols). If the answer does involve surds, then factorization wasn't the right approach to take. This *completing the*

[19]solving our quadratic equal to zero

square method does get a little trickier if the starting equation doesn't begin with $1x^2$, so here's a full typical example.

3.9.2 Completing the square: a typical difficult example

We are asked to solve

$$3x^2 + 2x - 7 = 0 \tag{3.20}$$

We first need to get rid of the factor of 3, so we divide through by 3 (because we need $1x^2$ on the front not $3x^2$) and get

$$x^2 + \frac{2}{3}x - \frac{7}{3} = 0$$

Now we need to find half of the x-term, i.e. half of $\frac{2}{3}$, which is $\frac{1}{3}$ and form the squaring bracket. We want these two sides to balance:

$$\left(x + \frac{1}{3}\right)^2 + \mathrm{Y} = x^2 + \frac{2}{3}x - \frac{7}{3}$$

where we need to work out what value to place in for Y. The x^2 and x terms already match, by our clever construction, so it's only the constant parts that need to be matched still[20].

The easiest way to find Y is just to copy the existing constant term down, and subtract off the extra final term you've introduced with your brackets expansion. i.e.

$$\left(x + \frac{1}{3}\right)^2 - \frac{7}{3} - \left(\frac{1}{3}\right)^2.$$

We have copied out the $-\frac{7}{3}$ from the original, but we have introduced a $\left(\frac{1}{3}\right)^2$ with our bracket, so we just subtract it off again.
The reason this works is because

$$\left(x + \frac{1}{3}\right)^2 - \left(\frac{1}{3}\right)^2 = x^2 + \frac{2}{3}x,$$

by subtracting off the square of the constant term in our bracket we just get the x^2 and x terms.

So,

$$\begin{aligned} x^2 + \frac{2}{3}x - \frac{7}{3} &= \left(x + \frac{1}{3}\right)^2 - \left(\frac{1}{3}\right)^2 - \frac{7}{3} \\ &= \left(x + \frac{1}{3}\right)^2 - \frac{22}{9}, \end{aligned}$$

since $-\frac{1}{9} - \frac{7}{3} = -\frac{1}{9} - \frac{21}{9} = -\frac{22}{9}$.

So we've turned our original equation (3.20) into

$$\left(x + \frac{1}{3}\right)^2 - \frac{22}{9} = 0 \tag{3.21}$$

The final steps are to make x the subject.

- We shall add $\frac{22}{9}$ to be both sides, then
- Square root both sides (remember there are two answers)
- Subtract $\frac{1}{3}$ from both sides to turn the left from $x + \frac{1}{3}$ into just x

[20] by choosing Y

Let's see these steps in action:

$$\begin{aligned}\left(x+\frac{1}{3}\right)^2-\frac{22}{9}&=0\\ \left(x+\frac{1}{3}\right)^2&=\frac{22}{9}\\ x+\frac{1}{3}&=+\sqrt{\frac{22}{9}}\text{ or }-\sqrt{\frac{22}{9}}\\ x&=+\sqrt{\frac{22}{9}}-\frac{1}{3}\text{ or }-\sqrt{\frac{22}{9}}-\frac{1}{3}\end{aligned}$$

These are our two answers for x.

As mentioned at the start of this section, this method is a little more work than using the formula (see 3.10). In the long term it is worth knowing this method too, but do expect to make some algebraic errors when first trying to use it.

Here is a video demonstration of the method, for you to watch before attempting your own examples:

```
#> [Video appears here in html version
#>  the hyperlink is provided below]
```

Alternative direct video link:

https://gcu.planetestream.com/View.aspx?id=5991~4y~AogxJYFc

Practice Complete the square for these six quadratic equations:

$$x^2+4x+9=0$$

$$x^2+4x-12=0$$

$$x^2-5x+\frac{1}{4}=0$$

$$x^2-5x+3=0$$

$$2x^2-2x+1=0$$

$$x^2-x-1=0$$

Here are the six correct answers, in random order. Match your six answers up with these six formulae:

$$\left(x-\frac{1}{2}\right)^2+\frac{1}{4}=0$$

$$(x+2)^2+5=0$$

$$\left(x-\frac{5}{2}\right)^2-\frac{13}{4}=0$$

$$\left(x-\frac{1}{2}\right)^2-\frac{5}{4}=0$$

$$\left(x-\frac{5}{2}\right)^2-6=0$$

$$(x+2)^2-16=0$$

Bonus question: Which of these problems has solutions of $x=\frac{5}{2}+\sqrt{6}$, and $x=\frac{5}{2}-\sqrt{6}$?

3.10 The Quadratic formula

As explained in the Completing the Square section (3.9), the quadratic formula also always works. It has the advantage of being slightly faster, assuming you can remember the formula, and don't mess up any of the algebra.

To use the quadratic formula you first need to put your problem into the standard format:

The **standard quadratic equation format** is

$$Ax^2 + Bx + C = 0 \tag{3.22}$$

where A, B and C are constants to be identified.

Here are some examples for you to do and check your answers:

Quadratic equation	A	B	C
$x^2 - 3x + 2 = 0$	1	-3	2
$-x^2 + x + 7 = 0$	-1	1	7
$3x^2 - x + 1 = 0$	3	-1	1
$-5x^2 + 13 = 0$	-5	0	13
$x^2 + 4x = 0$	1	4	0

Once you're comfortable with identifying A, B and C you can use the quadratic formula.

The **Quadratic Formula** says that the solutions to a standard format quadratic are:

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \tag{3.23}$$

where $\pm$ means two answers, one with a $+$ and the other with a $-$.

Key points to note:

- It's $-B$ not $+B$ at the start of the top of the fraction.
- The entire top of the fraction is divided by $2A$ not just one of the parts.
- Take care with negative signs, especially in the $-4AC$ part. If AC is negative then $-4AC$ is positive.
- If $B^2 - 4AC$ is negative, there will be no *real* solutions.[21] See Solubility Section 3.11 for further explanation.
- If $B^2 - 4AC = 0$ then both the $\pm$ answers give the same answer, so there's only one distinct answer for x. It's sometimes called a repeated solution.
- Many authors call the quantity $B^2 - 4AC$ the *discriminant* of the quadratic equation.

So the full version of the two solutions are:

$$x = \frac{-B + \sqrt{B^2 - 4AC}}{2A}, \quad \text{and} \quad x = \frac{-B - \sqrt{B^2 - 4AC}}{2A}$$

[21] The word *real* is used since there will be solutions in complex numbers.

3.10.1 A simple example

Solve $3x^2 - x - 1 = 0$ using the quadratic formula.
Answer: We identify $A = 3$, $B = -1$ and $C = -1$. So we substitute to get:

$$x = \frac{- -1 + \sqrt{(-1)^2 - 4 \times 3 \times -1}}{6},$$

and also,

$$x = \frac{- -1 - \sqrt{(-1)^2 - 4 \times 3 \times -1}}{6}$$

which simplify into:

$$x = \frac{1 + \sqrt{1+12}}{6}, \quad \text{and} \quad x = \frac{1 - \sqrt{1+12}}{6}$$

and finally into

$$x = \frac{1 + \sqrt{13}}{6}, \quad \text{and} \quad x = \frac{1 - \sqrt{13}}{6}$$

3.10.2 A video demonstration

Finally in this section, a short video demonstration of the use of the formula, with commentary.

```
#> [Video appears here in html version
#>  the hyperlink is provided below]
```

Alternative direct video link: https://gcu.planetestream.com/View.aspx?id=3616~4q~qNPdenUP

Rather than provide more examples here, I suggest you search online for one of the many quadratic solving websites. It is good to get practice in how to search for ways to check your work, and many of them will also contain automated examples for you to try. At time of writing this link is a good example MathsIsFun.com

3.11 Solubility of quadratic equations

We discussed above that the method of factorization doesn't always work, but that the other two methods do always work. However, what we mean by *work* is only that they will find the solutions **if they exist**.

When we attempt to solve a quadratic equation, we are trying to solve an equation that can first be re-arranged into our standard format as:

$$Ax^2 + Bx + C = 0.$$

However, it's quite easy to construct an example with no solutions at all. Just select $A = 1$, $B = 0$ and $C = 1$ and we get

$$x^2 + 1 = 0.$$

There are at least two ways to see that this equation has no solutions:

- x^2, as a square, is never negative, so after adding 1 to it means the left-hand side of the equation cannot equal zero;
- The graph of $y = x^2 + 1$ is U-shaped but never touches the x-axis. But the x-axis is where $y = 0$, i.e. $x^2 + 1 = 0$

Warning:
Not all quadratic equations have solutions. If the graph of $y = Ax^2 + Bx + C$ lies entirely above, or entirely below, the x-axis and never touches it, then there will be no values of x where $Ax^2+Bx+C = 0$.
There are actually only three cases:

- The graph passes through the x-axis twice (this is typical);
- The trough/peak of the u-shaped/n-shaped graph exactly touches the x-axis (this is rare);
- The graph never goes through the x-axis (this is also fairly typical).

The first case, we get two solutions, and will be very typical of problems where you are asked to *Solve* or *Find the solutions of.*
The second case, you get one[a] solution. This may also occur in examples you are given, an example would be $(x - 1)^2 = 0$.
The third case, where there are no solutions[b], is not that unusual but if you're asked to find the solutions to a quadratic there is often an implicit implication that there are solutions to find. Similarly if the problem is derived from a real-life problem where solutions must exist, then this won't happen.

[a] technically we say *repeated*
[b] without learning about, and using, complex numbers

The good news, is that it's fairly easy to notice that you're working with one of these cases with no solutions, as long as you are *completing the square* or using the *quadratic formula*.

When completing the square, you will know this *no solutions* case has occurred if at the late stage of square-rooting both sides the right hand side is negative and so cannot be square-rooted.

When using the quadratic formula, the quantity you square root is $B^2 - 4AC$ (also called the *discriminant* in many notes). If this quantity is negative, then there will be no solutions.

If you know you're being expected to find actual solutions, but your completing the square requires square-rooting a negative number; or your $B^2 - 4AC$ is negative... then you have probably already made an algebraic mistake. Go back and carefully check your algebra.

3.12 Simultaneous linear equations

We have already seen *linear equations*, these were generally written in the format $y = A + Bx$, for some constants A and B. The word *simultaneous* refers to there being more than one equation, each of which needs to simultaneously [22] be true for a solution to be valid.

When solving simultaneous linear equations we generally start with a slightly different (but completely equivalent) display format, we move the unknown variables onto one side and put the constant on the other side. So rather than writing $y = 2x + 1$ we write $-2x + y = 1$ or $2x - y = -1$ both of which are the same and are just re-arrangements of the first. Two simultaneous equations might then look like this:

$$2x - y = -1 \tag{3.24}$$

$$x + 3y = 24 \tag{3.25}$$

To solve this pair of equations, we are looking for a pair of values x, y which simultaneously make both equations true. For example, $x = 2, y = 5$, which we write $(x, y) = (2, 5)$ solves (3.24) perfectly, because $2 \times 2 - 5 = -1$. However, this pair doesn't solve (3.25) because $2 + 3 \times 5 = 17 \neq 24$. Trial-and-error is almost never a good idea for trying to solve such equations, because there are so many possibilities.

The solution which works for both (3.24) and (3.25) at the same time is $(x, y) = (3, 7)$ but to find it will normally involve the use of one of the two standard techniques below:

[22] i.e. at the same time

- Elimination (Section 3.12.2), or
- Substitution (Section 3.12.3)

Before demonstrating the methods, we should first be totally clear what we mean by simultaneous equations.

The general format for a pair of simultaneous equations is written:

$$\begin{aligned} Ax + By &= E \\ Cx + Dy &= F \end{aligned}$$

where A, B, C, D, E and F are all constants.

The unknown variables don't need to be x and y, but we just choose them as convenient letters. Similarly, although there are positive signs in the equations above, any variables could be negative which would override those signs. Furthermore, you can also generalize this problem to solve more than two simultaneous equations. However, with three (or more) unknowns both of these methods begin to take a long time and other more advanced techniques which use matrices are highly recommended.

Though beyond the scope of this chapter, you can (try and) solve any number of simultaneous equations as long as

$$\text{The number of equations} = \text{The number of unknown variables}$$

For example, here are two problems with 3 and 4 unknowns respectively:

$$\begin{aligned} x + 2y + 5z &= 8 \\ 3x - y + z &= 3 \\ 3x + y - 7z &= 12 \end{aligned}$$

and

$$\begin{aligned} w + x + y + z &= 5 \\ 2w - x + 3y - z &= -6 \\ -3w + x + 2y - z &= -1 \\ 0.5w - 2x + 3y + 4z &= 6 \end{aligned}$$

Much like when solving quadratic equations the answers in real-life are generally not nice round integers, although practice examples often are chosen in order to make the algebra fairly clean and so they do have integer answers. The example presented in (3.24) and (3.25) above was chosen to have a nice integer answer.

Warning: We shall solve this example by the two standard methods below, and the answers will be nice round integers. However, the 'normal' situation with solving simultaneous equations with real-life numbers results in answers which are just boring decimal numbers (or fractions) and not nice integers.

3.12.1 The geometrical intepretation

Returning now to two unknowns, which we'll call x and y... before we dive into the main methods, a short geometrical interpretation.

Geometrical interpretation A linear equation in x and y actually describes all the points that make up a straight line on the standard x and y axes (i.e a graph).
So finding a pair of numbers (x, y) which simultaneously solve two linear equations, means you're finding co-ordinates (x, y) which **lie on both lines**. So you're actually finding the intersection of two lines.

3.12.2 The method of elimination

As the name suggests, the idea here is to *eliminate* something while solving the equations. The way this is done is by taking the two equations and adding or subtracting one from the other in a clever way to make one of the unknown letters cancel – and thus be eliminated.

We shall try and solve:

$$2x - y = -1 \tag{3.26}$$

$$x + 3y = 24 \tag{3.27}$$

It will be important to have labelled the equations so we can reference them in what follows.

There are two ways to do do this method,

- Either we choose to eliminate x, or
- We choose to eliminate y.

Either choice is fine, you normally just choose the one that will be easier to do, algebraically.

The idea is to add (or subtract) two equations together (adding the lefts and the rights) to form a new equation. If we do it immediately then both x and y will survive the process, on the left-hand-side. However, if we start by multiplying both sides of (3.26) by 3 then it turns into:

$$6x - 3y = -3 \tag{3.28}$$

This equation is just an equivalent re-written version of (3.26)[23]. However, we carefully chose to multiply by 3 because we can see it creates a $-3y$ in (3.28) which will cancel with the $+3y$ visible inside the other equation, (equation (3.27)).

So now we add together equations (3.27) and (3.28). This involves adding the left-hand-sides together and setting the answer equal to the sum of the right-hand-sides. We get:

$$(x + 3y) + (6x - 3y) = (24) + (-3) \tag{3.29}$$
$$7x = 21 \tag{3.30}$$

Notice that the $+3y - 3y = 0$ and $24 + (-3) = 21$.

And we have reached a linear equation that mentions only x which can be solved by usual algebraic methods. In this case it's very easy as we start with $7x = 21$ and dividing both sides by 7 immediately yields that $x = 3$.

This tell us that we need $x = 3$ to be true for both our original equations to be simultaneously true. But what about y? We can just use any of the equations to find the y which works with $x = 3$. All equations we use should give the same accompanying y, which in this case yields $y = 7$ and our only solution is

$$(x, y) = (3, 7) .$$

[23] which could be recovered by dividing both sides by 3

Alternatively, we could have chosen to eliminate x rather than y. In this case we notice that it appears as $+2x$ in (3.26) and as x in (3.27).
The clever step here would be to take the second equation, (3.27), and multiply both sides by 2 to create a new equation that also contains $2x$.
However, now since both x terms are the same (and not the negative of each other as with the y example above) we should then **subtract** one equation from the other. Then the x terms will disappear from the next equation. You should try this yourself and see if you discover that $y = 7$ immediately.

In both these examples it was possible to leave one equation alone and multiply the other to get ready for the addition/subtraction. In more complicated examples you might need to multiply both equations by something to get two equations with matching x or y terms.

3.12.3 The method of substitution

This method requires less thinking than the elimination method. In the elimination method we needed to make a clever choice for the multiplication of one equation, to ensure one of the variables cancelled when we then added or subtracted our equations.

In contrast, with substitution, such thought isn't required. However, the resulting calculations normally involve fractions or messier algebra.

With substitution the idea is as follows:

- Start with two equations as usual.
- Select one unknown variable, and one equation (which mentions it).
- Re-arrange the chosen equation to make the chosen variable the subject.
- Substitute the formula for your chosen variable into the other equation.
- Re-arrange and solve the for the variable you didn't select.

The first couple of steps above, do involve a choice, and with practice you might develop the skills to select the easier route[24]. But which variable and equation works most easily will depend upon the question and can be hard to predict.

We shall use the method to solve the same problem again:

$$2x - y = -1 \tag{3.31}$$

$$x + 3y = 24 \tag{3.32}$$

An easy choice is not difficult to spot here. We want to pick a variable and an equation then make our chosen variable the subject of our equation.

We choose the second equation ((3.32)) and the variable x. This is easy because it requires very little algebra to make x the subject. We begin with $x + 3y = 24$ and just need to subtract $3y$ from both sides to move the $3y$ to the other side and reach:

$$x = 24 - 3y \tag{3.33}$$

We now have a formula for our chosen variable, x, which we *substitute* into the other equation ((3.31)). By substitute we mean to use our known value of x (we just read the right-hand-side of (3.33)) inside the first equation.

Here is the result of the substitution and simplifying algebra:

[24] if one exists

$$\begin{aligned}
2x - y &= -1 \\
2(24 - 3y) - y &= -1 \\
48 - 7y &= -1 \\
48 &= -1 + 7y \\
49 &= 7y \\
7 &= y
\end{aligned}$$

We replaced/substituted x with $24 - 3y$ because we know they are equal. Then proceeded with standard algebra steps to make y the subject. Resulting in the discovery that $y = 7$ is the answer.

As in the method of elimination this gives us the value of one unknown variable, and we can use any of our equations to find the value of the other variable. In this case you can quickly discover when $y = 7$ that $x = 3$ and we obtain the same answer as previously.

Alternatively, a messier route tries to make y the subject of (3.32). The re-arrangement gives

$$y = \frac{24 - x}{3}.$$

This can be substituted into the other equation: with $2x - y = -1$ becoming

$$2x - \left(\frac{24 - x}{3}\right) = -1$$

but then the algebra required here to re-arrange and discover $x = 3$ is a fair bit messier. If you can avoid introducing fractions then it's definitely recommended, but not always possible.

There are two other combinations of subject variable and equation we could have made.
You could of course make y the subject of (3.31) and the calculation would be straightforward.
However, making x the subject of (3.31) leads to another messy calculation.

3.12.4 Special cases

This isn't quite the end of the story for simultaneous equations. When following either of the methods above there are two unexpected things that can happen. When you are re-arranging an equation to make a variable, say x, the subject either of these events could happen:

1. All the x-terms might cancel, and you reach a totally factual equation like $4 = 4$ or $0 = 0$,
2. All the x-terms might cancel, and you reach a totally incorrect equation like $4 = 2$ or $1 = 0$.

Both of these can occur[25].

In Case 1 above, you've reached an equation that works for any and all x so there are solutions for every x! There will be infinitely many answers. Geometrically (see Section 3.12.1) you will have started with two equations describing exactly the same line, so all points on the line solve both equations!

In Case 2 above, you've reached an equation for which there are no x values that can make it true. So there are no answers that work for x! Geometrically (see Section 3.12.1) you will have started with two parallel lines which never meet each other. There will be no simultaneous solutions to your two equations.

Here's a video to illustrate Case 1 (above) occurring:

```
#> [Video appears here in html version
#>  the hyperlink is provided below]
```

[25] though not at the same time!

Alternative direct video link:

https://gcu.planetestream.com/View.aspx?id=5992~4z~aYw9tYQz

Here's a very similar video, but this time it's Case 2 that occurs:

```
#> [Video appears here in html version
#>  the hyperlink is provided below]
```

Alternative direct video link:

https://gcu.planetestream.com/View.aspx?id=5993~4A~CLLKdXaJ

Here's is a random example to test yourself, you can click to try another different question after answering. I suggest you try both methods separately to see which one you prefer.

```
#> [Embedded question appears here in html version
#>  the hyperlink is provided below]
```

Link to Numbas on the web

In addition I recommend you look online for nice simultaneous equations solvers which can show you worked solutions to problems you try. At time of writing, this is a nice example: equationcalc.com

Finally some advanced examples, again you can randomize and get a new question via the link.

```
#> [Embedded question appears here in html version
#>  the hyperlink is provided below]
```

[Link to Numbas on the web](https://numbas.mathcentre.ac.uk/question/114447/advanced-simultaneous/embed/?token=0578e3c1-2599-47a3-9b3d-ae0b5cafd7df

3.13 Further practice exercises

For most topics in algebra the best way to learn and improve is through practice. So you are recommended to seek out your own further exercises.

However, there is also a specific further exercises section at the end of the notes here: Section 6.

3.14 Summary of chapter

Having worked through this chapter and attempted the exercises you should now have understanding of the following topics:

- Develop confidence working with basic operations, brackets, powers and grouping terms;
- Work with exponential and logarithm functions, including knowledge of the log laws;
- Develop confidence using algebra to simplify and solve equations, including the use of operation inverses;
- Be able to identify a quadratic equation and be able to use:
  - Factorization,
  - Completing the square, or
  - The Quadratic Formula to solve them; and
- Be able to solve simultaneous linear equations by elimination and substitution methods.

Chapter 4

Vectors

Vectors are useful mathematical objects for describing, simultaneously, a real-world quantity with **length and accompanying direction**. For travel purposes, a vector might describe moving North for 3 kilometres – the direction being North, and the length[1] being 3 kilometres. This scenario is an example of what is called a *displacement vector*: a vector which describes a physical displacement (a 'movement'). Vectors can also describe other real-world phenomena, especially in the area of mechanics, such as forces, velocity and even momentum. A proper description of a force involves the *size* or *strength* [2] of the force, together with a direction in which the force is acting. For example, when studying a mechanics problem with gravity the force of gravity will have a strength (in Newtons) and a direction (downwards).

Given their description, vectors can always be described geometrically by a line segment with an arrowhead. The length of the segment describes the length of the vector, and the way the line points (with the arrowhead) gives the direction. Here are some 2-dimensional examples, in Figure 4.1:

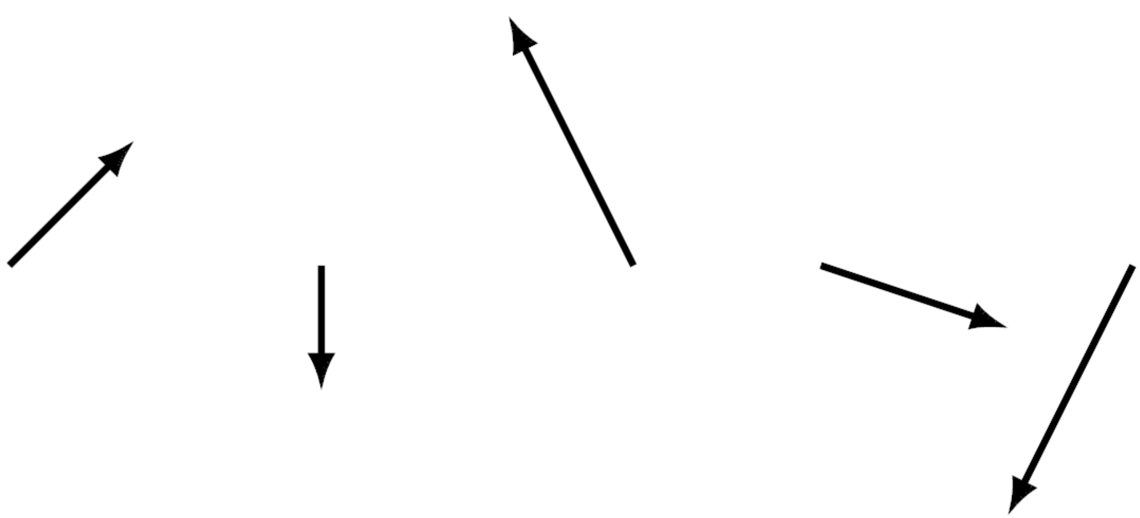

Figure 4.1: Some examples of vectors

Where a vector is drawn is not important, only its length and direction, so two vectors can be identical and just drawn in different locations. In Figure 4.2 are five vectors, they are all the same vector, just drawn in different places:

[1] also called *distance*, or *magnitude*
[2] or *magnitude*

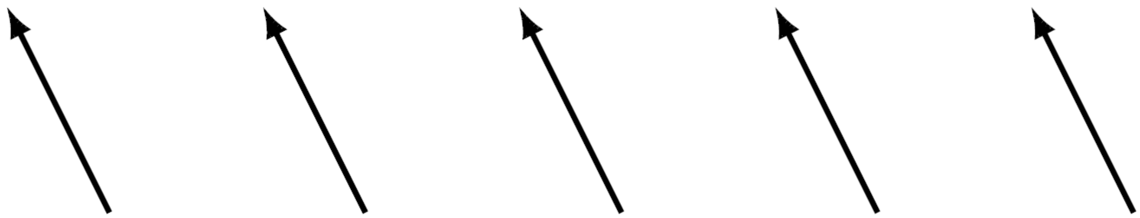

Figure 4.2: Five identical vectors

4.1 Scalars and vectors

Vectors differ from what are called *scalars* in one specific way, a *scalar* is just single number used to describe a length/size/strength/magnitude[3] and doesn't include a direction. Scalars are therefore nothing special at all, and just another name for any quantified physical quantity, like temperature, voltage, distance, height, mass, volume, density and many many more. All of these quantities are describable with a single value (and accompanying units) such as a voltage of 9 volts, a temperature of 21.4°C, a density of 450 kg m^{-3}, or mass of 74 kg. When performing algebra and calculations we often neglect the units (re-inserting them at the end) and so a scalar is just a number like 9 or 21.4.

Vectors, on the other hand additionally always include a direction, such as the example '3km North' earlier, a velocity like '45ms^{-1} East', or 'a force of 10N acting at an angle 30 degrees anticlockwise from the horizontal'.

Beware: when talking about vectors there a few quantities that have specific mathematical meanings, which can differ from how they are sometimes used in common English language.
The first common example concerns speed and velocity:

- Speed – this is a scalar and refers purely to how fast something is travelling.
- Velocity – this is a vector and tells you both speed and direction.

So it is mathematically wrong to say someone has velocity of 20 miles per hour, that is their *speed*.
The second common example concerns distance and displacement:

- Distance – this is a scalar and refers purely to how far apart two places are.
- Displacement – this is a vector and refers to the movement required between two places, so includes a distance and direction.

So it is mathematically wrong to say an object has experienced a displacement of 10 metres. You need to also specify in what direction it moved if you want to call it a displacement. Alternatively it would be fine to just say it has moved a distance of 10 metres, but this is less descriptive.

4.2 Notation for vectors

4.2.1 Fonts for vectors

Mathematically it's very tedious, not very productive, and pretty imprecise to have to refer to directions of vectors in terms of compass points like we have done above. It makes a lot more sense to agree some good notation instead.

We begin with the first good notation. If we are talking about a vector and want to give it an algebraic name like v (note this is the English letter v in italic maths font[4]) then we underline the letter, so we in fact write $\underline{v}$ which signifies that it represents a vector not a scalar.

This approach can make it easier to read equations which contain vectors and scalars. The bad news for students is that it's not universally agreed as a notation, and some authors just use bold, and so write $\mathbf{v}$ instead. Yet more authors instead place a tiny arrow on top of the letter, and write $\vec{v}$.

For consistency we shall try and use just the underline notation in these notes, so we'll write $\underline{a}$, $\underline{b}$, $\underline{u}$, $\underline{v}$ etc...

[3] other words exist too
[4] one common problem when first working with vectors is similarity of the letters u and v in maths fonts

4.2.2 Mathematical representation

The more interesting question is how to actually describe the length and direction in a clean, and easy way. The answer is that a number of good methods have been invented, and the best way generally depends upon the context of what you wish to do with your vector. Consequently there are a few totally equivalent but different-looking ways to represent a vector. We shall discuss the different representations or formats in later sections.

For now we just introduce them by name:

- Polar format – here you describe the length and then the direction (as an angle from the horizontal).
- Rectangular/Cartesian format – here you use the good old standard (x, y) axes as your basis.
- Geometric format – here you draw an arrow of a particular length (not very useful except in hand-drawings).

In addition, inside the rectangular format we shall see there is a very convenient bracket notation used too, sometimes called matrix format.

4.3 Cartesian/Rectangular notation

We begin with probably the easiest to understand notation. We start with a vector like this one, which has length $\sqrt{13} \approx 3.606$ and makes an angle of 33.69 degrees, measured in the anticlockwise direction, with the horizontal (see Figure 4.3):

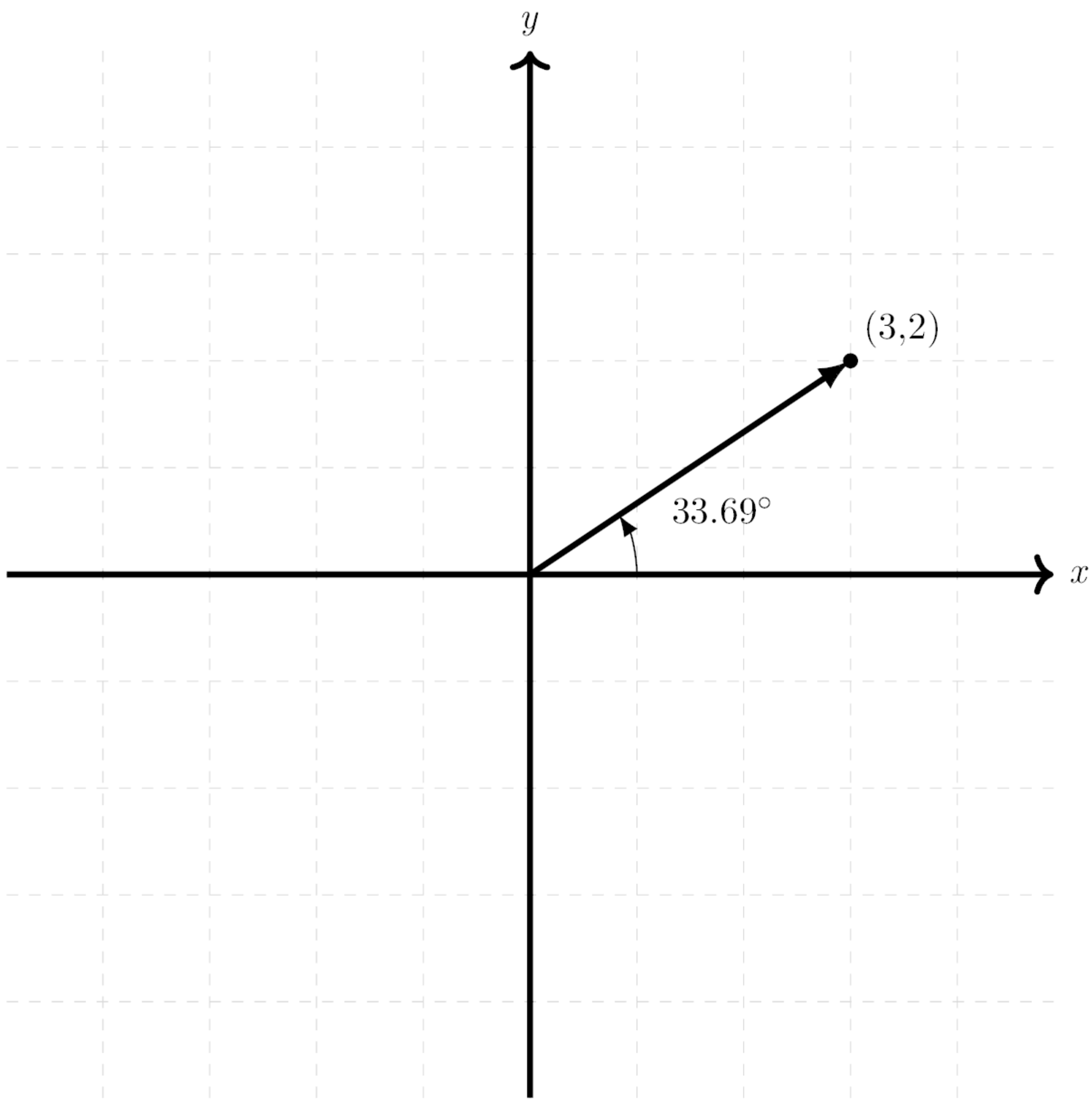

Figure 4.3: A vector of length $\sqrt{13} \approx 3.606$ at an angle of 33.69°.

The choice of $\sqrt{13}$ was special so that our vector represents a horizontal movement of $+3$ and vertical movement of $+2$. It is very typical for the length to be a surd[5], in fact it is to be expected if the horizontal and vertical lengths are both integers. You can use trigonometry (SOH-CAH-TOA) to check that $\tan(33.69^\circ) = \frac{2}{3}$[6].

This vector can be described, rather than by the length and direction, by its horizontal and vertical movements. Note that it didn't need to be drawn starting at the origin, but it just made it easier to see the movements. The way we write this vector in rectangular format is as $(3, 2)$.

[5] the square root of an integer
[6] I rounded the angle for convenience

Rectangular (also called Cartesian[a]) format for a vector involves working out the horizontal and vertical movement represented by the vector's arrow. Then placing them in a bracket of format

$$(x, y)$$

where the first *component*, x, represents the horizontal movement, and the second component, y, represents the vertical component.

[a]named after the famous mathematician and philosopher Descartes, who is credited with inventing the x-y axis pictures

How to remember which way around to write the *components*?
A common way to remember which component is which is via this alphabet-based memory aid...
fi**R**st = **R**ows
se**C**ond = **C**olumns
Where **R**ows refers to horizontal movements, and **C**olumns refers to vertical movements.

Let's see a worked example:

Identify in the picture below the following five vectors: $(1, 2), (-3, -1), (2, 1), (-1, 3), (-3, 0)$ you may assume the dotted grid lines are spaced 1 unit apart.

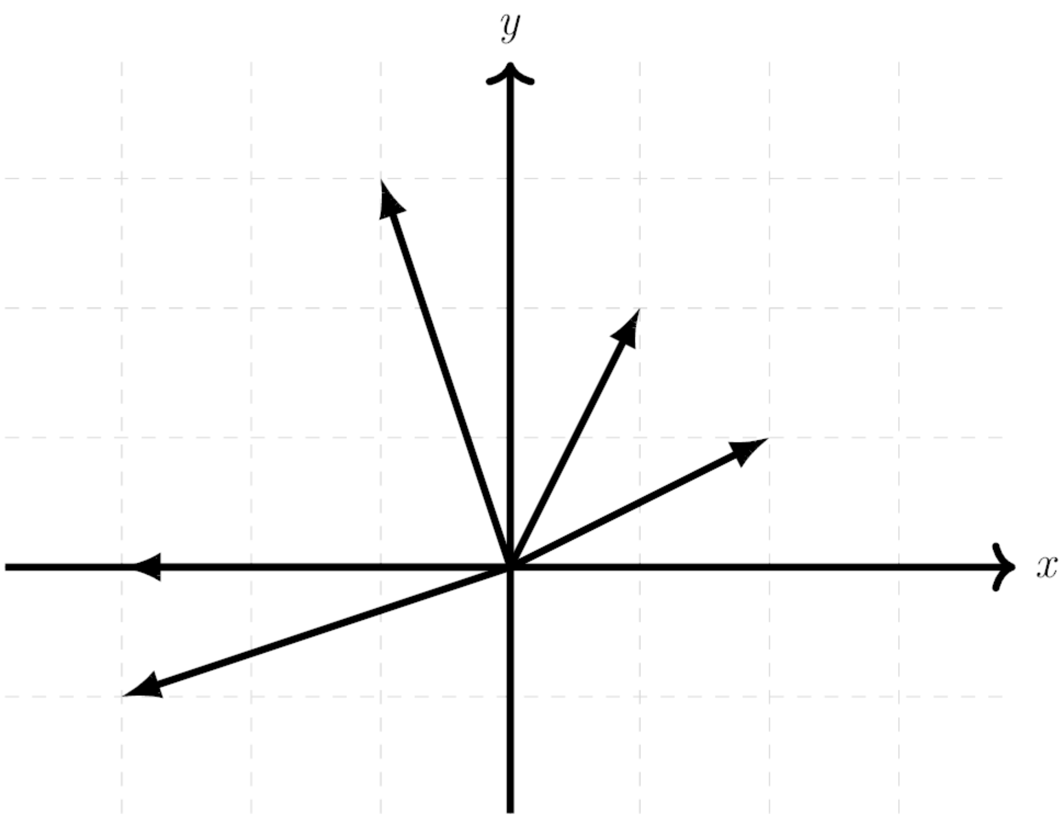

Figure 4.4: Five vectors for you to label using their rectangular format

The answers have been labelled on the picture below:

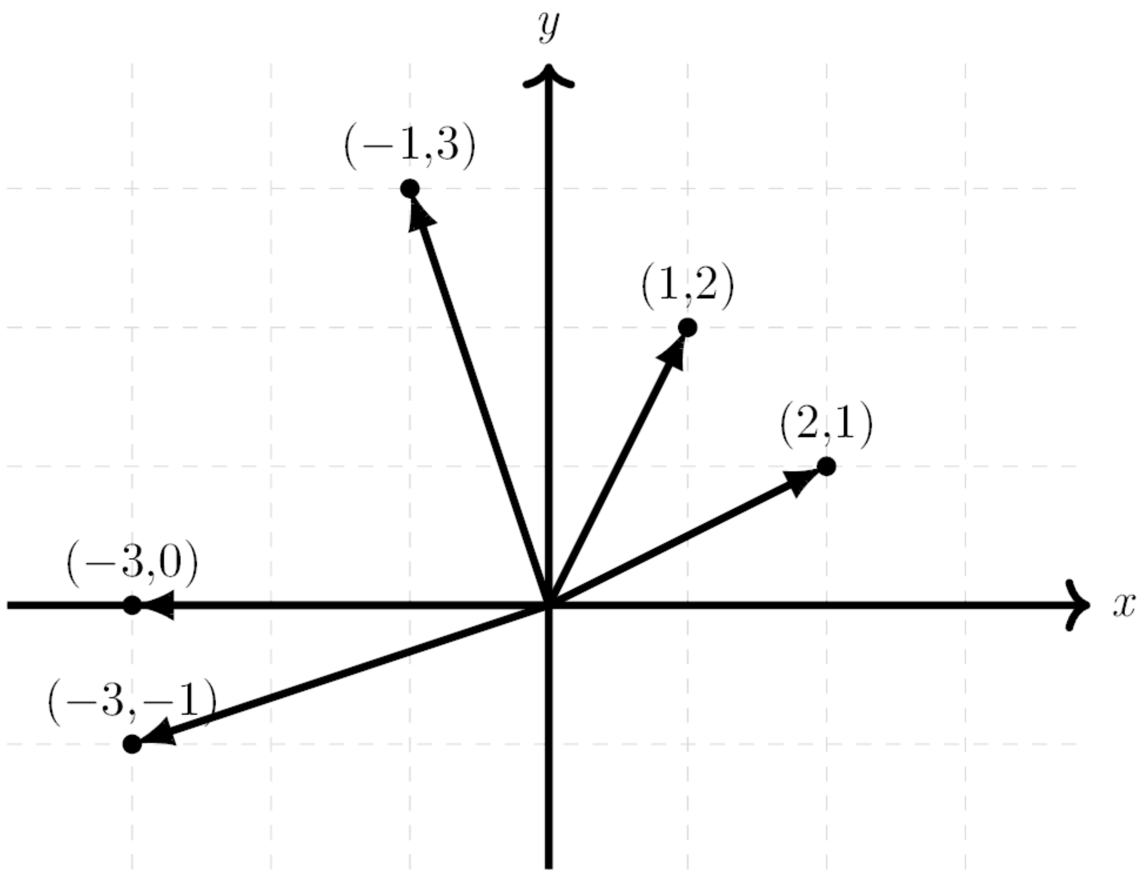

Figure 4.5: Five vectors labelled in rectangular format

There is sometimes confusion between this vector format and the x, y notation you've used in the past to identify points on a graph (remember the vector is really the line joining the origin to this point). For this reason, many people prefer to use what is called the *matrix* or *column vector* format which merely involves writing the vector vertically rather than horizontally on the page. *Matrix* (and it's plural *matrices*) is a specific mathematical term for a grid of numbers inside brackets – matrices are covered in separate chapter.

Here are the five vectors, $(2, 1), (1, 2), (-1, 3), (-3, -1), (-3, 0)$, from our example above written in *matrix* format:

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \end{pmatrix}, \begin{pmatrix} -3 \\ -1 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \end{pmatrix}.$$

This format takes up more space in writing, but does have the benefit of being obviously a vector rather than describing a point. We shall later see that there is another format to write rectangular notation by introducing some base vectors called $\underline{i}$ and $\underline{j}$, but we won't use them yet.

4.4 Polar notation

Polar notation is perhaps the easiest notation to understand, but unfortunately isn't very useful in a lot of algebra applications. This notation fits nicely with our original geometric introduction of vectors, here is a summary.

The **polar** format of a vector involves drawing the vector starting at the origin on a graph and describing two quantities:

- the **length** of the vector

- the **angle** the vector makes anticlockwise with the positive x-axis

The length is normally called r, and is always chosen to be positive[7].

The angle is normally called θ (the Greek letter *theta*) and we shall choose to describe it as between -180° and $+180^\circ$. This means we don't write 200° but we write -160° instead[8]. This range of values is really just another notational choice, so you'll find some authors use the range of 0° to $+360^\circ$ for θ. Such authors will use 200° rather than -160°.

The notation itself is written either as (r, θ) or $r\angle\theta$, with the former being more popular. You do need to remember that the first number is always the length and the second number the angle. However, if you include the degree symbol, $^\circ$, it will be obvious.

Figure 4.6 is the same basic example seen in the introduction of the rectangular notation:

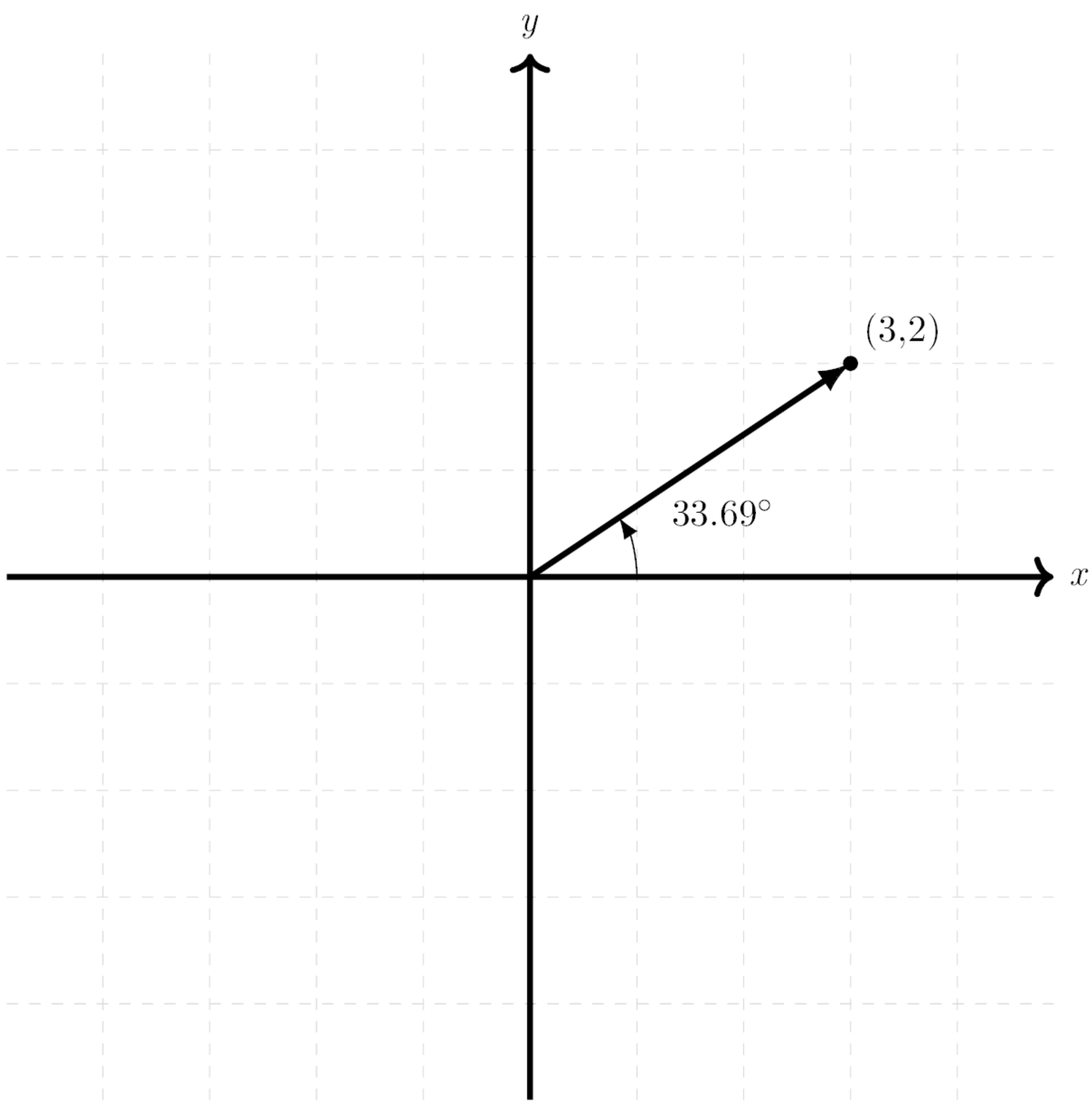

Figure 4.6: A vector of length $\sqrt{13} \approx 3.606$ at an angle of 33.69°.

This vector is easy to describe in polar notation as $(r, \theta) = (\sqrt{13}, 33.69^\circ)$

Figure 4.7 shows three more vectors that can be described via their length and angle.

[7]technically it might be zero
[8]they both describe the same angle really

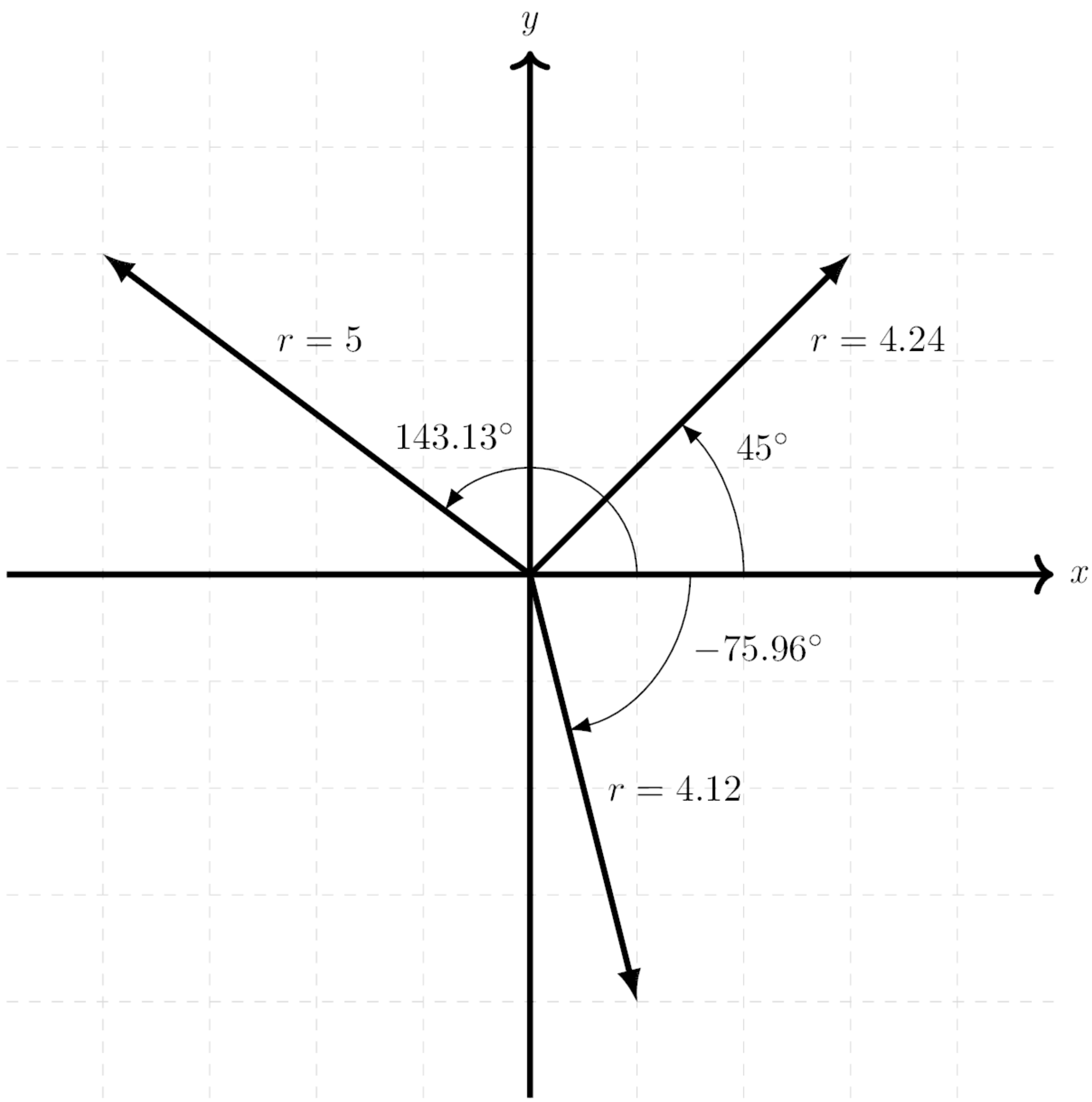

Figure 4.7: Three vectors with their angles and lengths labelled.

Notice the angle of the vector is always measured starting from the positive x-axis (this is called the pole, as in polar).

Notice also that the vector which points downwards has a negative angle, because we measure anticlockwise angles as positive, and clockwise as negative, with the positive positive x-axis measured as zero.

Here is a practice example:

Identify in the picture below the following three vectors written in polar notation: $(1.41, 45°), (5, 126.87°), (3.61, -123.69°)$ you may assume the dotted grid lines are spaced 1 unit apart.

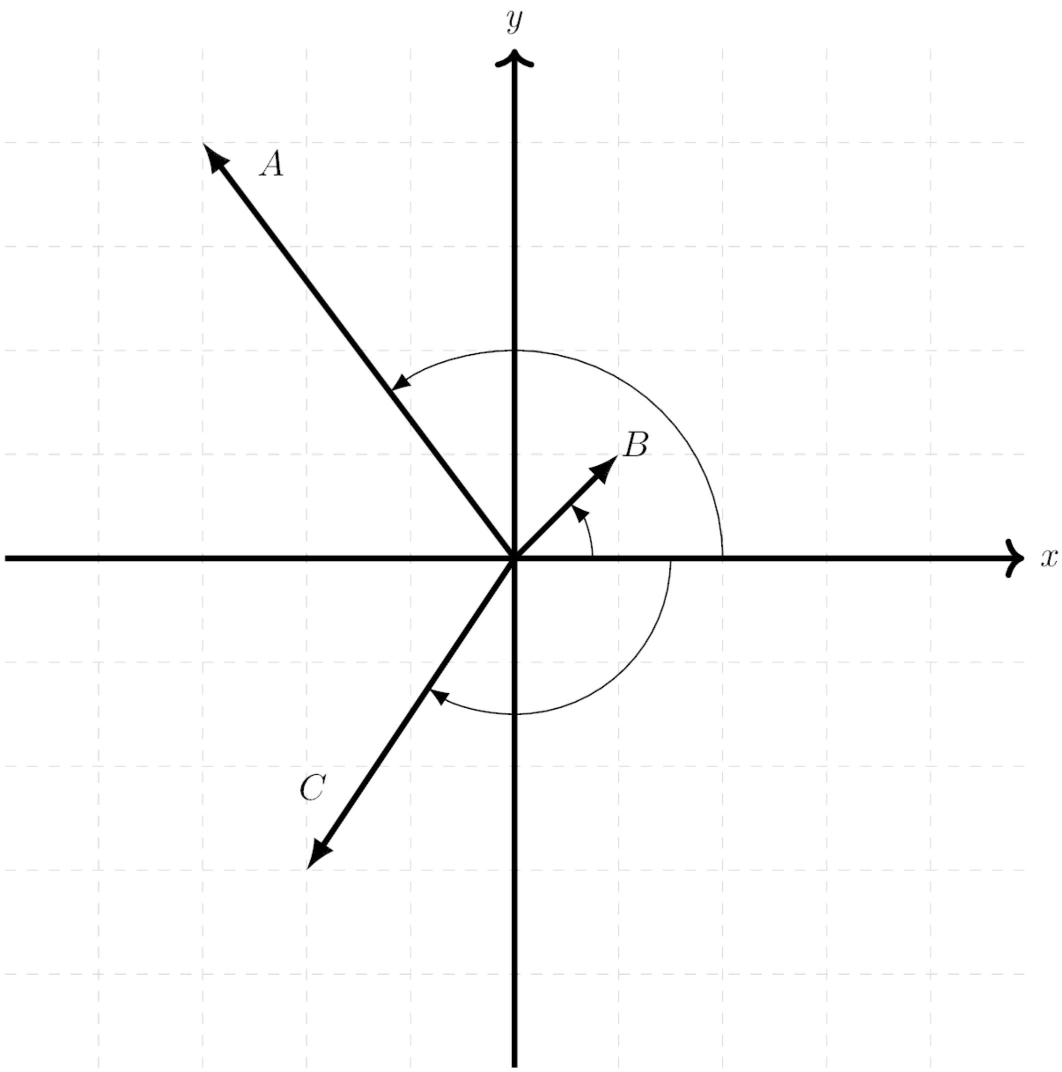

Figure 4.8: Three vectors labelled A, B and C.

The answer is in this[a] footnote.

[a] In the order they were written, they are B then A then C

4.5 Converting between Cartesian and Polar notations

As you may have already noticed there should be a fairly straightforward way to convert forwards and backwards between polar and rectangular notations. Clearly they are both valid ways to describe a vector so every vector should have both a polar and rectangular representation.

In order to convert between the two formats you need to use a little trigonometry, and Pythagoras' theorem.

Let's consider a vector $\underline{v}$ which we write in both notations. So,

- Polar notation: $\underline{v} = (r, \theta)$, and also

- Rectangular notation: $\underline{v} = \begin{pmatrix} x \\ y \end{pmatrix}$

4.5.1 Polar to Rectangular

The easiest direction for conversion is probably Polar to Rectangular. In Figure 4.9 below you can see that a vector with labelled polar notation (r, θ) has been drawn. Additionally a right-angled triangle has been drawn underneath this vector.

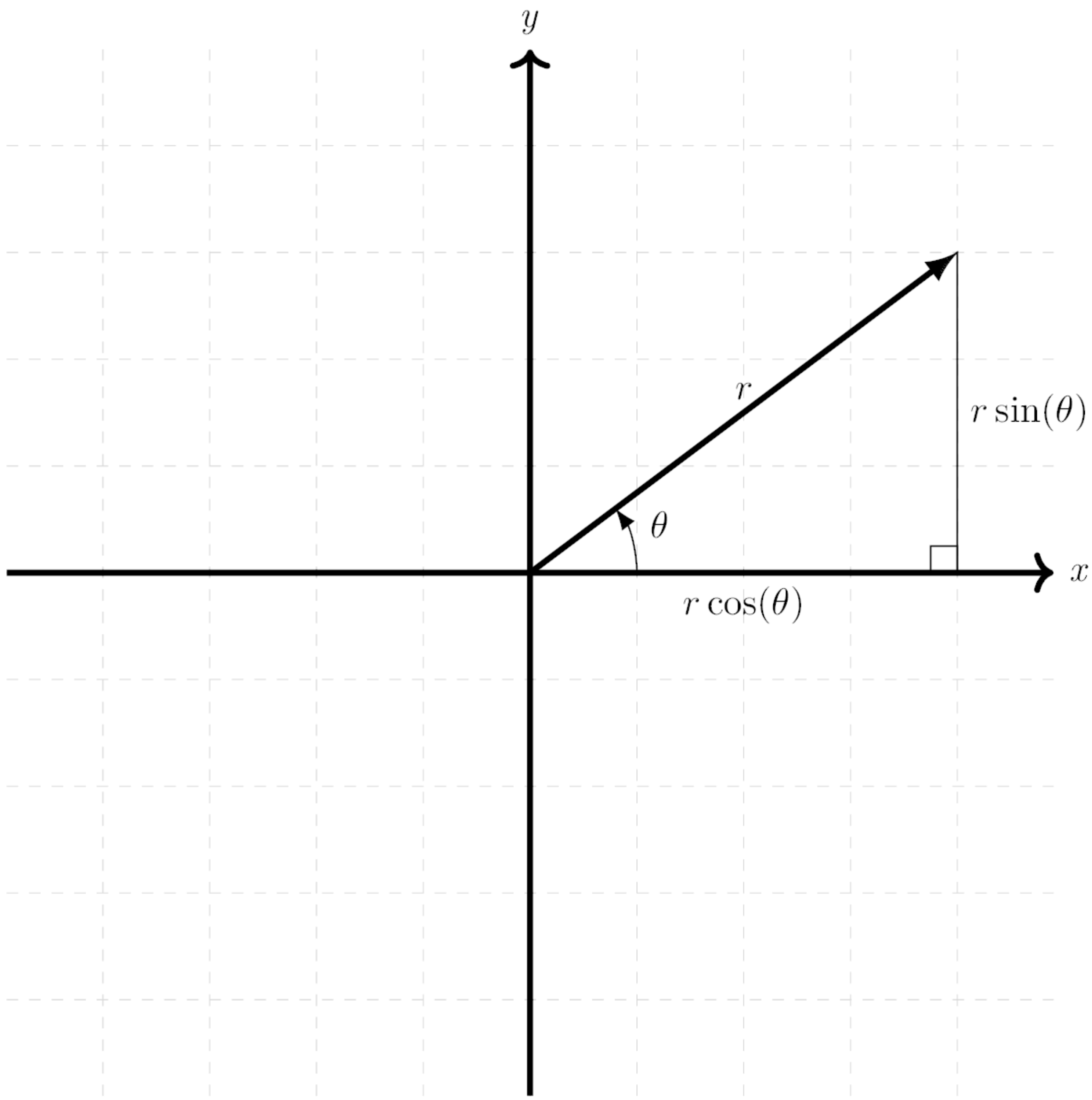

Figure 4.9: How to convert polar to rectangular notation

Basic trigonometry has been performed in the above diagram to label the x and y lengths of the triangle drawn. This means there is a straightforward conversion formula. If you would like to see the trigonometry behind this then here is a video, otherwise you can skip to the formula below.

```
#> [Video appears here in html version
#>  the hyperlink is provided below]
```

Alternative direct video link:

https://gcu.planetestream.com/View.aspx?id=5994~4B~dmaaP9m7

Standard conversion formula:

$$\begin{array}{cc} \text{(Polar notation)} & (r,\theta) \\ & \downarrow \\ \text{(Rectangular notation)} & (r\cos(\theta), r\sin(\theta)) \end{array} \tag{4.1}$$

This picture method works if the vector lives in the top-right quadrant on the graph, but what happens if the angle θ isn't between 0° and 90°?

The great news is that we don't need to worry, as these formulae with $\cos(\theta)$ and $\sin(\theta)$ also work outside this range.

Here is a video to help you see this conversion working for the angle 120°:

```
#> [Video appears here in html version
#>  the hyperlink is provided below]
```

Alternative direct video link:

https://gcu.planetestream.com/View.aspx?id=5995~4C~FapLz8xu

In the video we had $\theta = 120^\circ$ and so, since $\cos(120^\circ) = -0.5$,

$$r\cos(\theta) = r\cos(120^\circ) = -0.5r$$

i.e. the x co-ordinate becomes negative, which is what was expected for an angle of 120°. These conversion formulae, using $r\cos(\theta)$ and $r\sin(\theta)$, work for all r and all θ.

4.5.2 Rectangular to Polar

This conversion is also not super difficult, but it is a little trickier to make sure you get the angle correct.

We imagine this time starting with a vector with rectangular notation (x, y) and want to calculate its polar version (r, θ).

The answer is actually very easy if $x = 0$ because then the vector either points vertically upwards[9] or vertically downwards[10] and the angle will be 90° or -90° respectively. The length r will just be the size[11] of y.

We can solve the cases where $y = 0$ very similarly (the angle will be 0° or 180°).

So we will only need the calculations below if x and y are not zero.

We start with a sketch from which its obvious we can use Pythagoras' theorem to find r (Figure 4.10):

[9] if y is positive
[10] if y is negative
[11] see discussion below

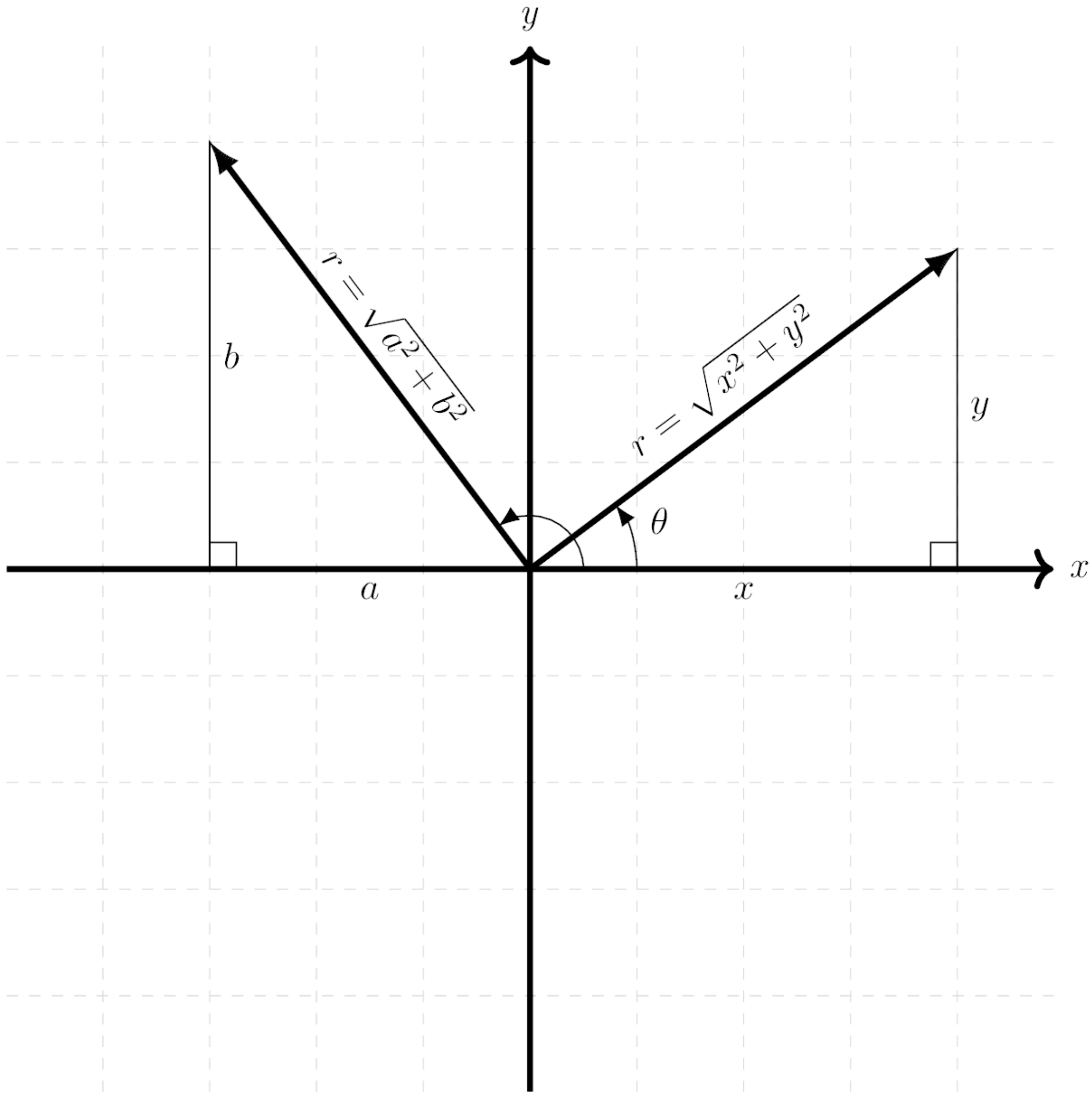

Figure 4.10: How to use Pythagoras to find vector length

Just looking at the vector lengths in this diagram it's clear that we can just square the two values, add them and then square root the answer to use Pythagoras to find the length of the vector. This even works in the case above where a is negative, because a^2 will be positive anyway.

Standard formula for polar length, r, from rectangular notation (x, y):

$$r = \sqrt{x^2 + y^2}$$

The difficulty comes with making sure we measure the correct angle in the diagram. Looking at the triangle drawn in the top-right, with side lengths x and y, we know the formula for the angle θ will be

$$\theta = \tan^{-1}\left(\frac{y}{x}\right), \quad \text{sometimes also called } \arctan\left(\frac{y}{x}\right).$$

However, calculating $\tan^{-1}\left(\frac{b}{a}\right)$ for the triangle in the top-left may not find the angle desired because most calculators give the answer to $\tan^{-1}$ that lies between $-90°$ and $90°$, and we want an angle that lies outside this range, at approximately $120°$. There are many valid approaches but perhaps the cleanest is to always convert the sides of the right-angled triangle into lengths (i.e. turn negative numbers positive) and consider this picture:

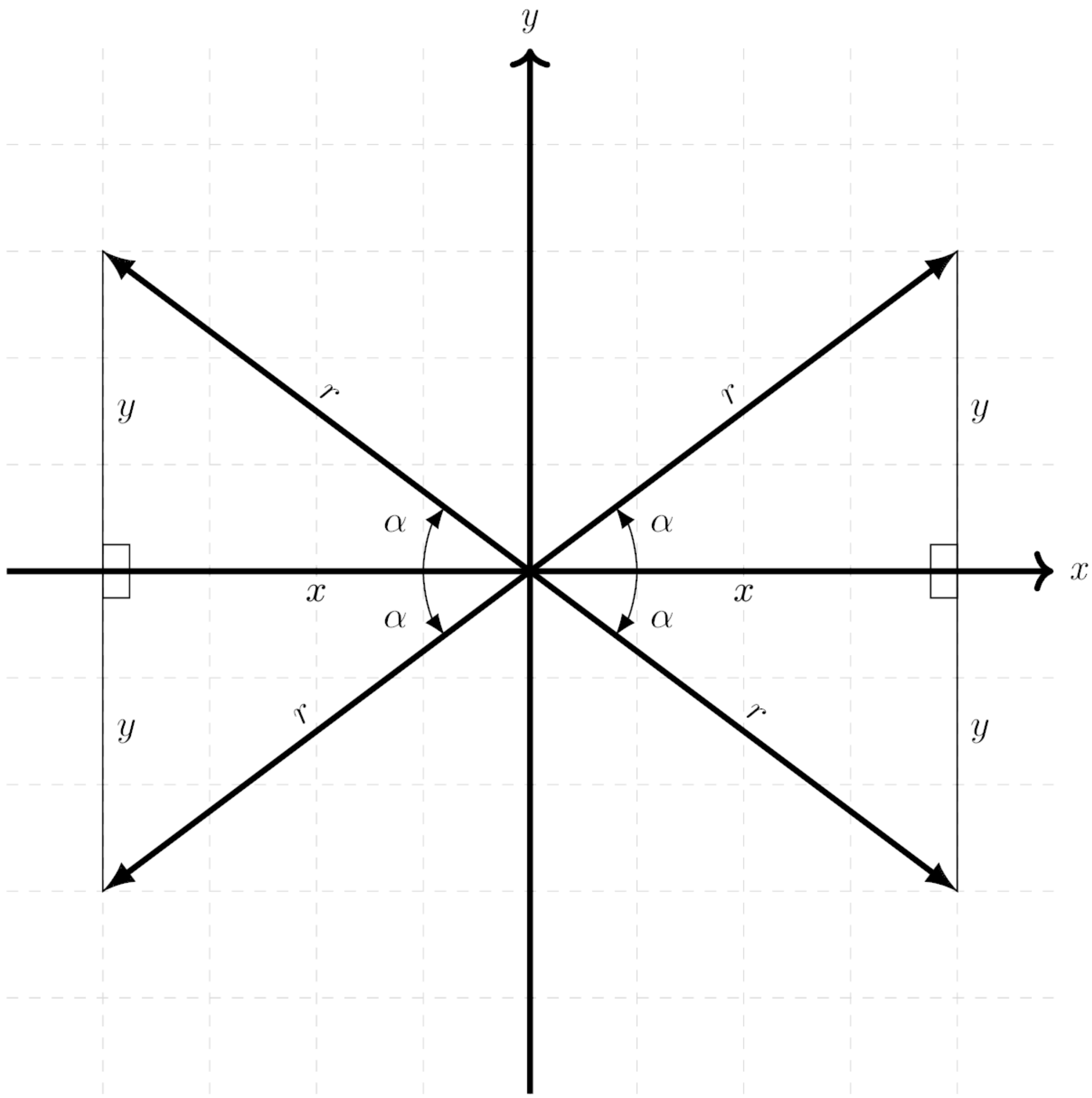

Figure 4.11: Using $\tan^{-1}\left(\frac{|y|}{|x|}\right)$ always returns one of these four positive α.

Notice in Figure 4.11 there are four vectors drawn, and four triangles (each with sides x and y) and that the angle labelled α is always the angle created between a vector and the x-axis. In this picture four identical angles are all labelled α and they are the positive angle obtained if you calculate

$$\alpha = \tan^{-1}\left(\frac{|y|}{|x|}\right)$$

where the notation $|y|$ and $|x|$ just means to take the size[12] of y and x respectively, basically just ignoring their sign. e.g. $|4| = 4, |-2| = 2, |1| = 1, |-17| = 17$ etc...

To work out the correct angle θ for the polar notation, isn't too difficult once you have α.

[12]formally called the absolute value

The formula for the polar angle θ from rectangular notation (x, y) can be found by first calculating

$$\alpha = \tan^{-1}\left(\frac{|y|}{|x|}\right)$$

Then, based on where the vector (starting from the origin) lives…

- In the top-right corner $\theta = \alpha$ already.
- In the bottom-right corner $\theta = -\alpha$,
- In the bottom-left corner $\theta = -180^\circ + \alpha$,
- In the top-left corner $\theta = 180^\circ - \alpha$.

It's easiest to just draw a sketch like Figure 4.11 above, to recreate these rather than try and memorize them.
To work these out manually just remember that the negative x-axis is 180° around from the positive x-axis.

```
#> [Embedded question appears here in html version
#>  the hyperlink is provided below]
```

Link to Numbas on the web

```
#> [Embedded question appears here in html version
#>  the hyperlink is provided below]
```

Link to Numbas on the web

4.6 Vector addition and subtraction

Now that we've invented a whole new type of object, a vector, and we've covered in detail the ways to represent vectors, we are allowed to decide what it means to add two vectors together, or subtract two vectors from each other. Luckily for us, mathematicians who first worked with vectors have already agreed upon a very useful meaning which allows us to solve a wide range of problems without getting too complicated.

If we recall that we can think of vectors as describing movements, then *adding two vectors* represents doing one movement after the other.

So if vector $\underline{v} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and vector $\underline{w} = \begin{pmatrix} -5 \\ 4 \end{pmatrix}$ so that

- $\underline{v}$ represents moving right 2 and down 3, and
- $\underline{w}$ represents moving left 5 and up 4

then we say that $\underline{v} + \underline{w}$ represents

'moving right 2 and down 3' **then** 'moving left 5 and up 4'.

Geometrically it can be calculated by drawing the vector $\underline{w}$ to start at the end of vector $\underline{v}$ like the picture in Figure 4.12. Notice that the vector $\underline{w}$ has been drawn to start at the end of $\underline{v}$.

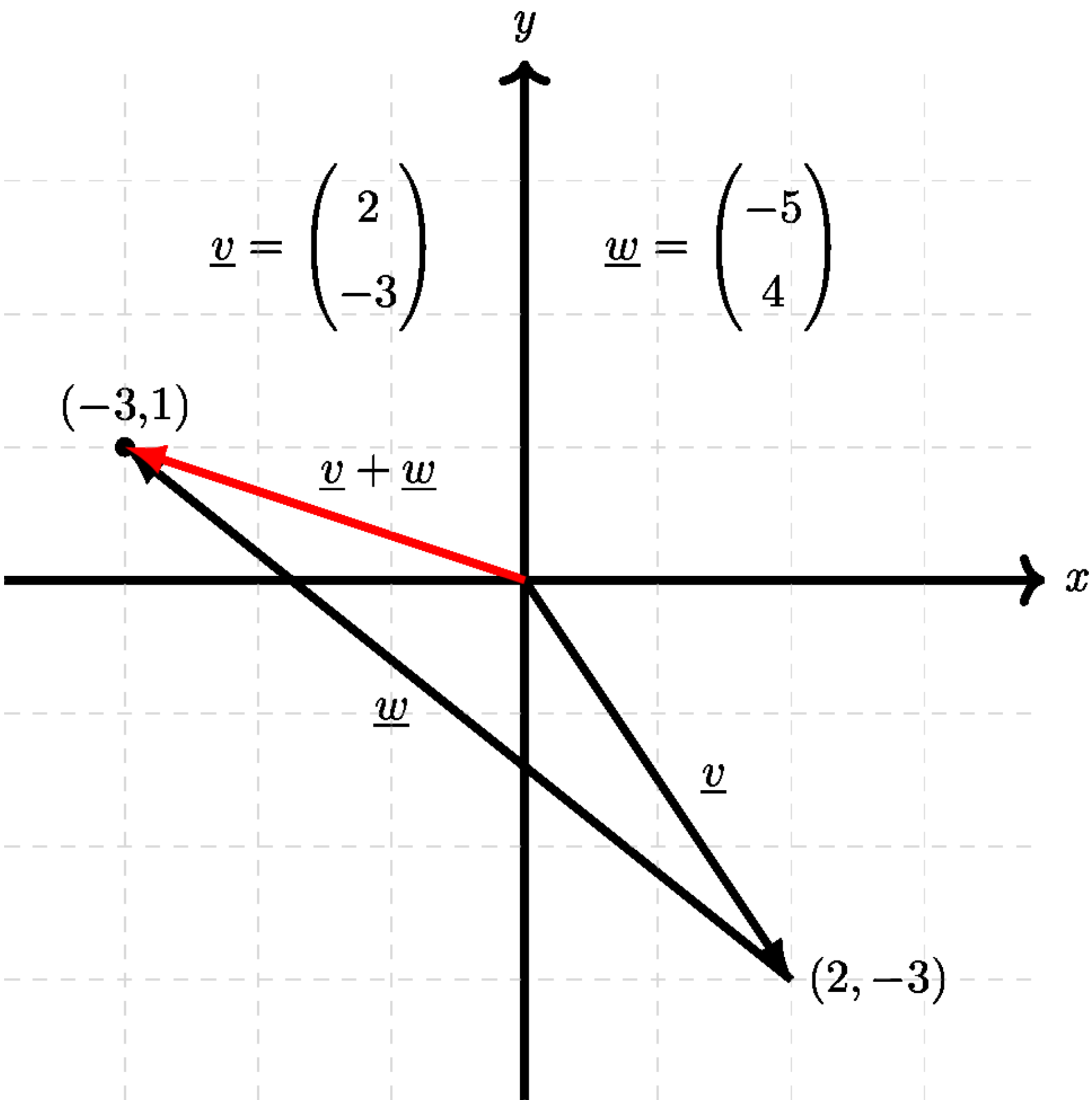

Figure 4.12: Illustration of geometry of vector addition.

Thankfully, algebraically it's even easier, if the vectors are in rectangular notation, you just add the corresponding *components* like this:

$$
\begin{aligned}
\underline{v} + \underline{w} &= \begin{pmatrix} 2 \\ -3 \end{pmatrix} + \begin{pmatrix} -5 \\ 4 \end{pmatrix} \\
&= \begin{pmatrix} (2) + (-5) \\ (-3) + (4) \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}
\end{aligned}
$$

So $\underline{v} + \underline{w}$ is the vector which represents moving 3 left and 1 up, i.e. the net effect of doing $\underline{v}$ and then $\underline{w}$[13].

[13]or the other way round too!

So the general formula for adding vectors just looks like this:

$$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix} \tag{4.2}$$

Subtraction works in exactly the same way, you can either think of $\underline{v} - \underline{w}$ as $\underline{v} + (-\underline{w})$ or use the formula:

$$\begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a-c \\ b-d \end{pmatrix} \tag{4.3}$$

If you begin with your vectors in polar notation, however, then it's much trickier to add them together. Indeed the easiest method is normally to convert them first into another notation (like rectangular) and then perform the calculation.

The examples below can be used to practice adding and subtracting vectors. You can generate a new random example using the button below each question. In this example, the *Reveal answers button* will also show you how to do the question if you are stuck.

```
#> [Embedded question appears here in html version
#>  the hyperlink is provided below]
```

Link to Numbas on the web

```
#> [Embedded question appears here in html version
#>  the hyperlink is provided below]
```

Link to Numbas on the web

4.7 Multiplying a vector by a scalar

Here we're combining two terms we have learnt in this chapter. Recall that a scalar is just another name for a constant, a single-value or number (in contrast to a vector which includes a direction). So if we have a vector $\underline{w}$ then examples of multiplying by a scalar would include

$$2\underline{w}, 3\underline{w}, -\underline{w}, 3.4\underline{w}, -\sqrt{7}w$$

Note, that in the third example, just like when performing algebra normally $-\underline{w}$ is just a short version of $-1 \times \underline{w}$.

Since the idea of multiplication is really just repeated addition, so that multiplying something by 2 is the same as adding it to itself, it's not a surprise that multiplying a vector by a scalar is also pretty easy.

If you've been wondering until now why scalars are called scalar… then you're about to see. Multiplication by scalars has the effect of *scaling* the vector, in the sense that *scaling* is used to change the size of an image. Scaling by a factor of 2 means to double the size, scaling by 3 means to triple the size, etc…

As a simple example, if $\underline{w} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ then $5\underline{w} = \begin{pmatrix} -5 \\ 10 \end{pmatrix}$.

The standard formula for scalar multiplication looks as follows. If we call our scalar k then in rectangular format

$$k \times \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} k \times x \\ k \times y \end{pmatrix} \tag{4.4}$$

Geometrically, it's also quite easy to see how scaling works. First, in Figure 4.13, some examples to show how scaling works for simple positive multiples.

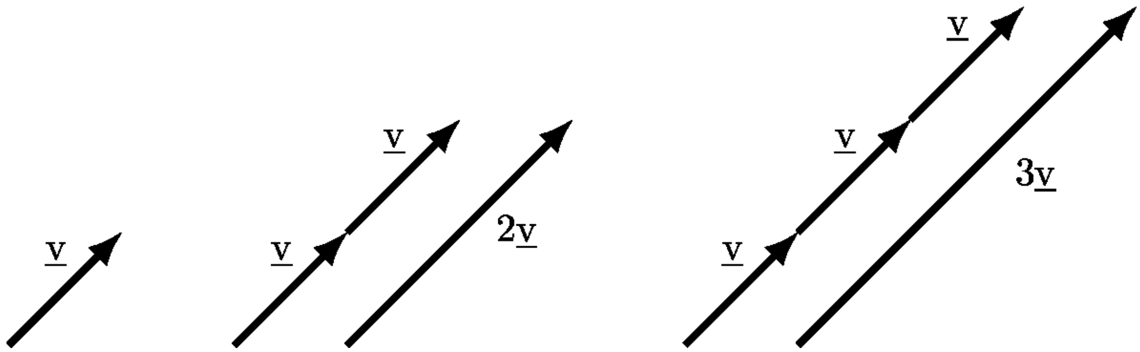

Figure 4.13: Illustration of geometry of vector addition.

Next, in Figure 4.14, we see that the negative of a vector just reverses the direction, and fractional multiples behave as expected. So $1.5\underline{v}$ is 50% longer than $\underline{v}$.

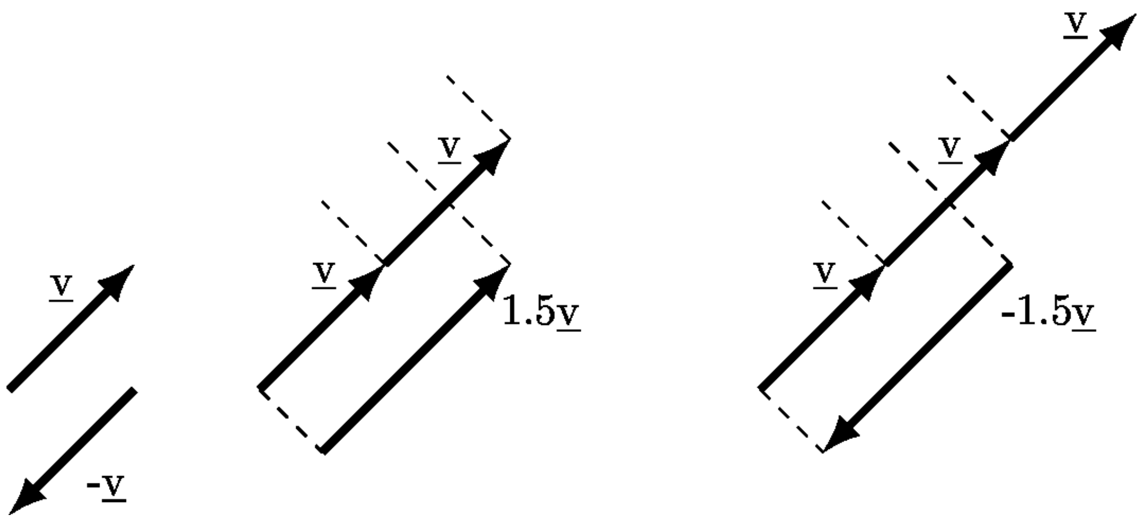

Figure 4.14: Illustration of geometry of scalar multiplication of a vector.

In polar notation it's also fairly easy, since scaling a vector never actually changes the direction of the vector, **except** when we multiply by a negative scalar. In which case the direction is completely reversed. Thus a vector with angle $\theta = 30^{\circ}$ when multiplied by -1 is completely reversed and now has angle -150°. Or a vector that begins with angle $\theta = -50^{\circ}$ is totally reversed and becomes a new vector with an angle of 130°. The precise formula depends on whether the original θ was negative or positive. Notice also that the length of a vector is doubled when it is multiplied by 2 or is multiplied by -2, it's just the angle also changes in the latter.

Multiplying vectors by scalars in polar form, is best visualized geometrically, but the formulae also look like this:
If $k > 0$ then

$$k \times (r, \theta) = (k \times r, \theta).$$

But if $k < 0$, then first recall that $-k = |k|$ is the positive scaling factor, and

$$k \times (r, \theta) = (|k| \times r, \theta + 180^{\circ} \text{ or } \theta - 180^{\circ})$$

which one of the two angles above is correct, is whichever one fits in the standard range between -180° and 180°.
It's not recommended to try and memorize this formula, best to get a geometric understanding by trying some examples.

Practice

In Figure 4.15, which of the four labelled vectors represents $\underline{v}$ and which one represents $-1 \times \underline{v}$, i.e. $-\underline{v}$?

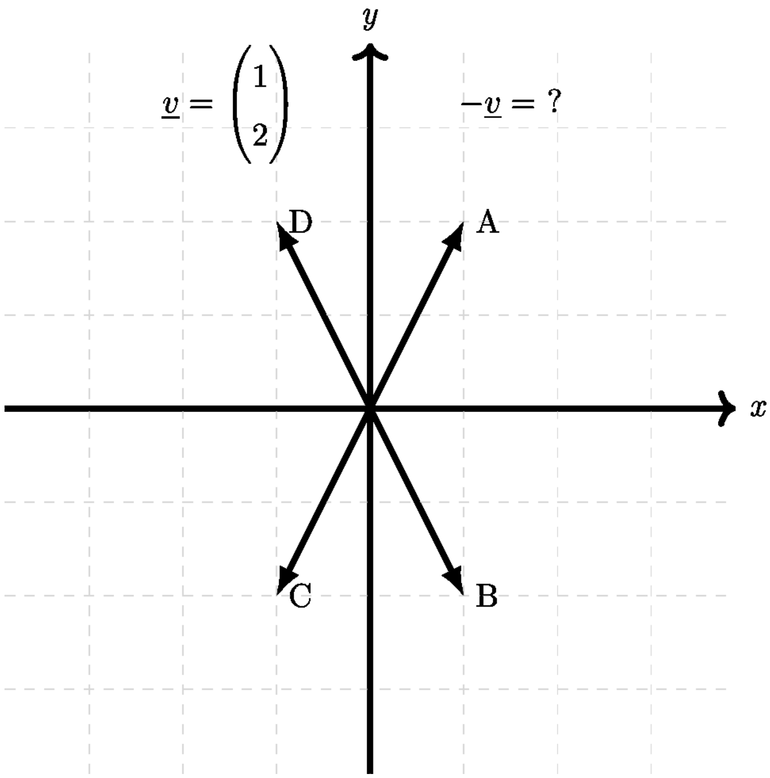

Figure 4.15: Four vectors, but which is $-\underline{v}$?

Next, a scaled vector. The (shortest) black vector is $\underline{v}$, which of the other vectors is $3\underline{v}$?

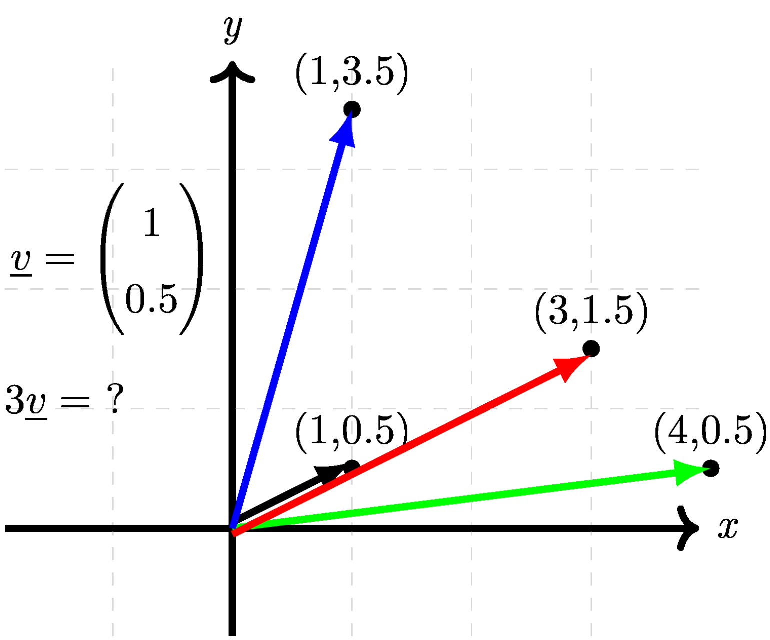

Figure 4.16: Which one is $3\underline{v}$?

Answers[a]

[a] In the first picture, $\underline{v}$ was A and $-\underline{v}$ was C. In the second picture it's the red line to $(3, 1.5)$ which depicts $3\underline{v}$.

4.8 Relative position vectors

One natural use of vectors is to represent locations of objects in a diagram, like in force diagrams. In such cases objects are often described by labelling their locations and the edges between such points can be represented as vectors.

A vector that describes the displacement between two positions in space is called a **relative position vector**.
e.g. if a point P is located at $(3, 1)$ and point Q is located at $(-1, 3)$ then, the position vector *of* Q *relative to* P is written $\overrightarrow{PQ}$ and calculated by subtracting the co-ordinates of P from Q. Alternatively, you can subtract $\overrightarrow{OP}$ from $\overrightarrow{OQ}$ like this:

$$\overrightarrow{PQ} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}.$$

The $\overrightarrow{PQ}$ notation is used, because a directed line (starting at P) joining P to Q will match the vector in direction and length.
Recall: this arrow notation to signify a vector is one of the alternatives to underlining a vector. It is almost always used in this kind of example where you have locations described by letters.

As previously, it is not important to learn the formula for the relative position vector, it is better for future applications if you can work it out by remembering that the relative position vector of Q from P is the vector required by someone who starts at P who wants to travel to Q.

In the examples above $P = (3, 1)$ and $Q = (-1, 3)$ so to reach Q from P requires:

- Subtracting 4 from the x-coordinate (notice $-4 = (-1) - (3)$), and
- Adding 2 to the y-coordinate (notice $2 = (3) - (1)$).

Here is that same example, shown in geometric form, where it is easier to see the vector $\overrightarrow{PQ}$:

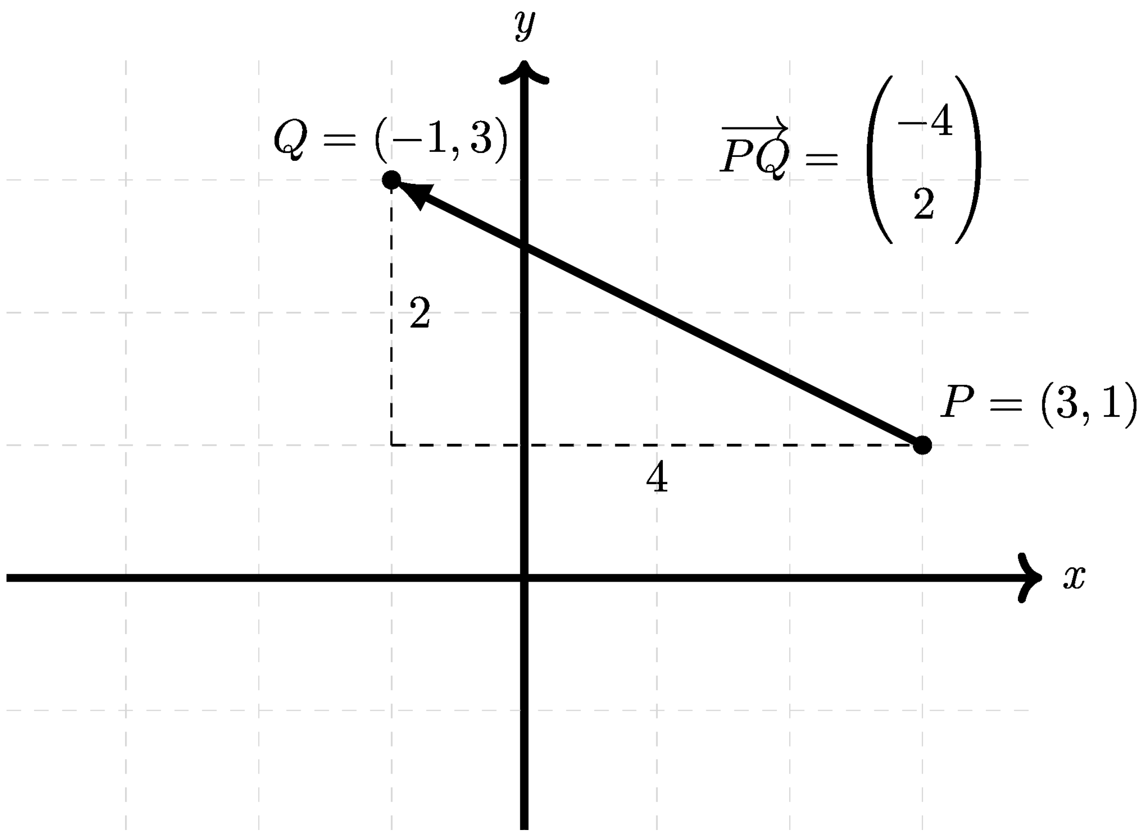

Figure 4.17: Illustration of a relative position vector. Here, the position vector of Q relative to P is written $\overrightarrow{PQ}$.

It is again hopefully clear that the calculation of a relative position vector is quite straightforward when vectors are written in rectangular format. Since this calculation is a subtraction the difficulty in doing this with vectors written in polar notation is the same as it was in Section 4.6.

4.9 The $\underline{i}$, $\underline{j}$ notation for vectors

Introduction of a final alternative rectangular notation has been saved until now as it really is just a completely equivalent alternative to the previously introduced Rectangular/Cartesian format.

Definition of the $\underline{i}$, $\underline{j}$ vectors.
For this notation we introduce two special vectors:

$$\underline{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \underline{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We have already introduced the standard rules for addition, subtraction and scalar multiplication of vectors. So it only remains for us to appreciate how these rules manifest in this new $\underline{i}$, $\underline{j}$ notation. One reason this notation is popular is because it allows the $\underline{i}$ and $\underline{j}$ to be treated much like we treat unknowns when we perform algebra.

Firstly, it's very easy to convert from rectangular to $\underline{i}$, $\underline{j}$ notation:

$$\begin{pmatrix} 3 \\ 7 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 7\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 3\underline{i} + 7\underline{j}$$

or in general

$$\begin{pmatrix} x \\ y \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix} = x\underline{i} + y\underline{j}$$

for any values (positive, zero, or negative) of x and y.

- The constant in front of the $\underline{i}$ vector corresponds to the x-coordinate.
- The constant in front of the $\underline{j}$ vector corresponds to the y-coordinate.

You can do addition and subtraction using this notation in a standard algebraic way, by grouping the $\underline{i}$ terms and $\underline{j}$ terms separately, just as you would two different letters in algebra.

For example,

$$\left(2\underline{i} + 3\underline{j}\right) + \left(3\underline{i} - 9\underline{j}\right) = 5\underline{i} - 6\underline{j}$$

or if $\underline{v} = 3\underline{i} + \underline{j}$ and $\underline{w} = -\underline{i} + 7\underline{j}$ then

$$\begin{aligned} 2\underline{v} - 3\underline{w} &= 2\left(3\underline{i} + \underline{j}\right) - 3\left(-\underline{i} + 7\underline{j}\right) \\ &= \left(6\underline{i} + 2\underline{j}\right) - \left(-3\underline{i} + 21\underline{j}\right) \\ &= 9\underline{i} - 19\underline{j} \end{aligned}$$

This notation is very often used in textbooks and resources as a way to avoid the slightly more space-consuming rectangular column vector notation which can span multiple lines of text. However, obviously some users may also just prefer it.

Practice
Match up these five vectors written in matrix notation with the $\underline{i}, \underline{j}$ vectors below.
The column vector formats:

i. $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

ii. $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

iii. $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$

iv. $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$

v. $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$

The $\underline{i}, \underline{j}$ notation:

a. $4\underline{i} + 3\underline{j}$
b. $2\underline{i}$
c. $3\underline{i} + 4\underline{j}$
d. $4\underline{i} - 3\underline{j}$
e. $2\underline{j}$

Answers[a]

[a](i)(c), (ii)(e), (iii)(b), (iv)(a), (v)(d)

4.10 The Scalar (or Dot) product of two vectors

Having discussed multiplying vectors by scalars in Section 4.7, one could naturally ask the question of whether vectors can also be multiplied by each other.

Since vectors were a new mathematical object, the creators were free to decide what this multiplication is designed to represent and mean. The result of the study was that there are actually two different useful ways to define the multiplication of one vector with another vector. This section (Section 4.10) will discuss the first of these:

- The Scalar Product (also known as the Dot ($\cdot$) Product);
- The Vector Product (also known as the Cross ($\times$) Product) will be discussed in Section 4.13

Each of these products has two names(!) because **one name describes what sort of output you get**, and the alternative name **describes the symbol used to denote the multiplication**. We cannot just use $\times$ like we have always done in algebra, because we are going to have two different ways to define multiplication.

Unsurprisingly, the Dot Product will use a dot like this, $\cdot$, and the Cross Product (see later) will use a cross, like this, $\times$. But because their other names are more helpful we shall stick to calling them the Scalar[14] and Vector[15] products for the time being.

[14]Dot
[15]Cross

The Scalar Product is a way to multiply two vectors which gives a scalar as its output. The easy formula definition available in rectangular notation is as follows.

If $\underline{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\underline{w} = \begin{pmatrix} a \\ b \end{pmatrix}$ then

$$\underline{v} \cdot \underline{w} = xa + yb$$

Let's see a few examples involving numbers, from which it should be obvious how to calculate the Scalar Product.

For vectors $\underline{v} = \begin{pmatrix} 8 \\ -2 \end{pmatrix}$ and $\underline{w} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ their Scalar Product is

$$\underline{v} \cdot \underline{w} = \begin{pmatrix} 8 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 8 \times 3 + (-2) \times 4 = 16.$$

For vectors $\underline{a} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} -2 \\ 7 \end{pmatrix}$ their Scalar Product is

$$\underline{a} \cdot \underline{b} = \begin{pmatrix} -1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 7 \end{pmatrix} = (-1) \times (-2) + 4 \times 7 = 30.$$

For vectors $\underline{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\underline{v}_2 = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ their Scalar Product is

$$\underline{v}_1 \cdot \underline{v}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 3 \end{pmatrix} = 3 \times (-1) + 1 \times 3 = 0.$$

It's worth noting in this final example we have used subscripts rather than different letters to name our vectors. This is a very typical approach because when working with vectors you can often quickly run out of new letters, so it's easier to use just one letter like $\underline{v}$ and use subscripts to name them differently.

There is an alternative definition of the Scalar Product, which is useful for finding the angle between two vectors, but not very useful for actually calculating the Dot Product:

$$\underline{v} \cdot \underline{w} = \text{length}(\underline{v}) \times \text{length}(\underline{w}) \times \cos(\theta) \tag{4.5}$$

where θ is the angle between the vectors $\underline{v}$ and $\underline{w}$.
Authors often use the notation $|\underline{v}|$ for length($\underline{v}$) so this formula is often written:

$$\underline{v} \cdot \underline{w} = |\underline{v}||\underline{w}| \cos(\theta). \tag{4.6}$$

If we have written our vector in Polar format then the length is just called r. If we have our vector in Rectangular notation then we need to use Pythagoras to work out its length.

As stated above, this formula is **not** generally useful for actually finding the Scalar Product of two vectors because the formula given before is so easy to use. Additionally we don't normally know the angle between two vectors. However, as we shall see in Section 4.11 we can use this new formula to work out the angle between two vectors. By the angle between two vectors, we just mean the angle you create if you draw both vectors starting at the same point and measure the gap from the first to the second vector, e.g.

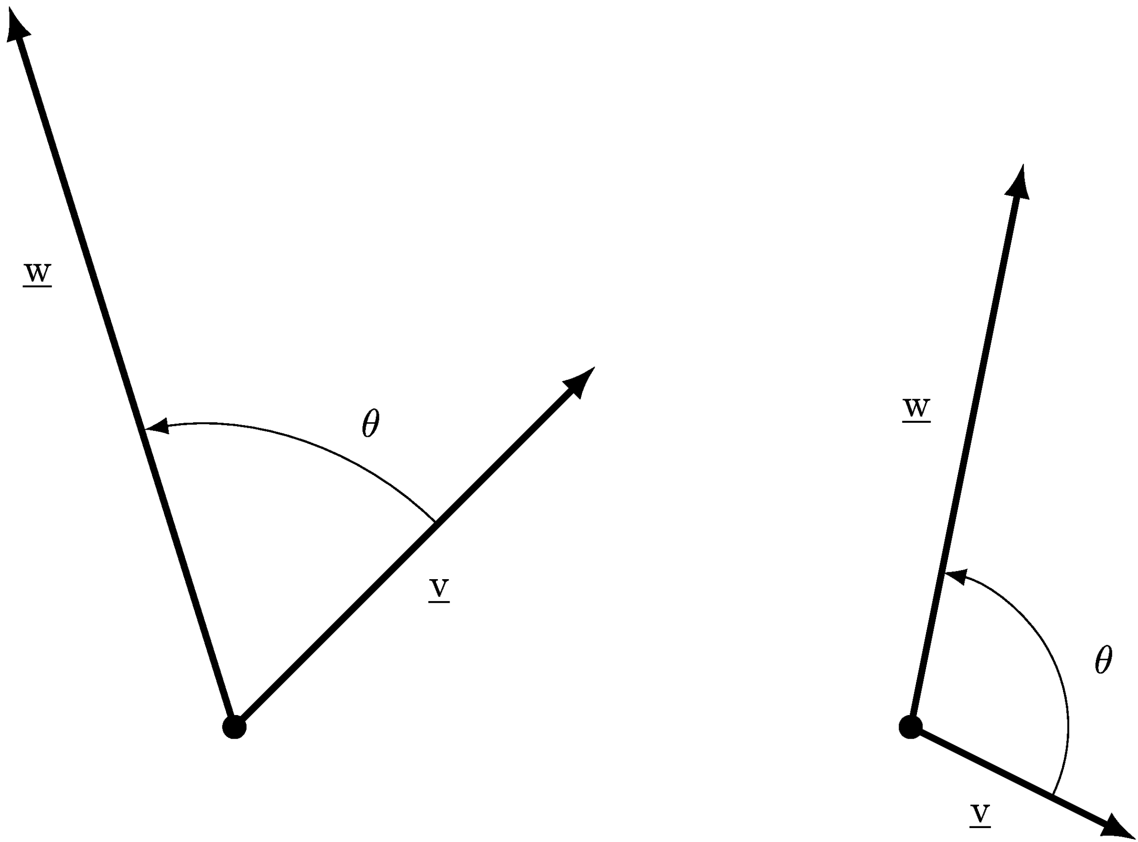

Figure 4.18: Two examples of the angle between two vectors, called θ in the Scalar Product formula.

As a final aside, while you will generally use the Rectangular format to work out a Scalar Product, if you do happen to start with two vectors in Polar format, e.g. $\underline{v} = (r_1, \theta_1)$ and $\underline{w} = (r_2, \theta_2)$ then finding the Scalar Product is actually pretty easy. We know that length($\underline{v}$) $= r_1$ and length($\underline{w}$) $= r_2$. And the angle between the two vectors will simply be $\theta_2 - \theta_1$ (draw a little sketch if you want). So, occasionally this version of the formula is useful:

$$(r_1, \theta_1) \cdot (r_2, \theta_2) = r_1 r_2 \cos(\theta_2 - \theta_1).$$

In the next section we shall discover why the Scalar Product is so useful.

Some examples to practice calculating Scalar Products will appear in the examples at the end of the notes in Section 7.

4.11 Applications of the Scalar Product

The Scalar Product calculation has many useful applications, below we shall discuss just four of these, although there are others.

4.11.1 How to find the angle between two vectors

Given two vectors, $\underline{v}$ and $\underline{w}$, we can use the Scalar Product to quickly find the angle between them. If you imagine drawing the two vectors both starting from the same point (see Figure 4.18 for two such examples) this is the angle we shall try to find.

There are four steps:

- Starting in Rectangular format, use the simple Scalar Product formula to calculate $\underline{v} \cdot \underline{w}$ (see (4.10));

- Find the lengths of the vectors (using Pythagoras);
- Re-arrange the $\underline{v} \cdot \underline{w} = \text{length}(\underline{v}) \times \text{length}(\underline{w}) \times \cos(\theta)$ formula to find $\cos(\theta)$;
- Use inverse cosine to find θ.

Let's try it with $\underline{v} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\underline{w} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$

First,

$$\underline{v} \cdot \underline{w} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 5 \end{pmatrix} = 2 \times 1 + (-1) \times 5 = -3.$$

Next we find the vectors' lengths,

$$\text{length}(\underline{v}) = |\underline{v}| = \sqrt{2^2 + (-1)^2} = \sqrt{5},$$

and

$$\text{length}(\underline{w}) = |\underline{w}| = \sqrt{1^2 + 5^2} = \sqrt{26}.$$

Substituting these values into either of the formulae above ((4.5) or (4.6)),

$$-3 = \sqrt{5}\sqrt{26}\cos(\theta),$$

which can be rearranged to say

$$\cos(\theta) = \frac{-3}{\sqrt{5}\sqrt{26}} \approx -0.2631.$$

Finally we use $\cos^{-1}$ to get

$$\theta = \cos^{-1}(-0.2631) \approx 105.255^\circ.$$

This is our angle from $\underline{v}$ to $\underline{w}$. Try sketching the vectors yourself and verify that the angle is indeed slightly more than a right-angle. In fact, this example is the right-most example in Figure 4.18!
A minor note is that the angle $360^\circ - \theta$ is always the angle between the vectors if you consider $\underline{w}$ to $\underline{v}$ rather than $\underline{v}$ to $\underline{w}$.

4.11.2 How to check if two vectors form a right-angle

If two vectors form a right-angle then the technical term used is that they are *orthogonal* (to each other)[16].

Checking for *orthogonality* is actually much easier than finding the angle between two vectors. Recall that if the angle $\theta = 90^\circ$ then

$$\cos(\theta) = \cos(90^\circ) = 0.$$

So if we look at the complicated Scalar Product formula ((4.5)):

$$\underline{v} \cdot \underline{w} = \text{length}(\underline{v}) \times \text{length}(\underline{w}) \times \cos(\theta),$$

we notice that if the angle between vectors $\underline{v}$ and $\underline{w}$ is $\theta = 90^\circ$ then $\cos(\theta) = 0$ and the right-hand side of the equation will equal 0! So it follows that $\underline{v} \cdot \underline{w} = 0$.

This means that whenever two vectors form a 90° angle, then their Scalar Product will be zero!

Even better is the news that by looking at the right-hand side of (4.11.2) the **only** way for the right-hand side to be equal to zero is if $\cos(\theta) = 0$ unless we've stupidly started with a vector of length 0 (which doesn't

[16] it's also called *perpendicular*

really make any angles)[17]. So as long as neither of our vectors is just a vector of zeroes, then the **only** way to get $\underline{v} \cdot \underline{w} = 0$ is via an angle of 90° (or 270°, if being picky, but that's just 90° in disguise).

Our conclusion is that to check if two vectors are orthogonal, you just find their Scalar Product and check if it's zero.

So the following two statements are actually saying exactly the same thing:

- $\underline{v} \cdot \underline{w} = 0$
- $\underline{v}$ and $\underline{w}$ form a right-angle.

We can do a quick couple of examples:

Let's try it with $\underline{v} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\underline{w} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$

$$\underline{v} \cdot \underline{w} = 2 \times 1 + (-1) \times 5 = -3 \neq 0.$$

So they are **not orthogonal**[a].

On the other hand, if we try $\underline{a} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$, then

$$\underline{a} \cdot \underline{b} = 3 \times (-1) + 1 \times 3 = 0$$

so these two vectors **are orthogonal**. Sketch them and see it for yourself if you wish.

[a] we actually just saw the angle they form is about 105°

4.11.3 Finding the projection/component of a vector in a particular direction

Many applications of vectors to real-world force diagrams involve identifying what effect a known force has upon an object. Forces have both a strength[18] and a direction, however, and this direction can strongly effect how much of a force's full strength acts on an object.

As an example, if you strike a nail with a hammer by hitting it directly flat on the head, then all of the force is acting in the direction of pushing the nail into your target. If, however, you strike the nail with an identical strength force, but only hit the nail at a 45° angle then some of your force is acting sideways and not assisting in pushing the nail into your target. That sideways force often results in you bending the nail, oops!

Another famous example is that of an object on a slope, sliding down under gravity. The steeper the slope the more of the gravitational force is aligned with the slope, and the shallower the slope the less effect the gravitational force has to move the object downhill. This process of calculating how much of the force acts in different directions is called *resolving*.

First **a method you don't need to learn for these notes**, the standard trigonometry approach. Here is a typical diagram for an object on a slope with a labelled angle and gravitational force:

[17] You may want to think about this.
[18] magnitude or length

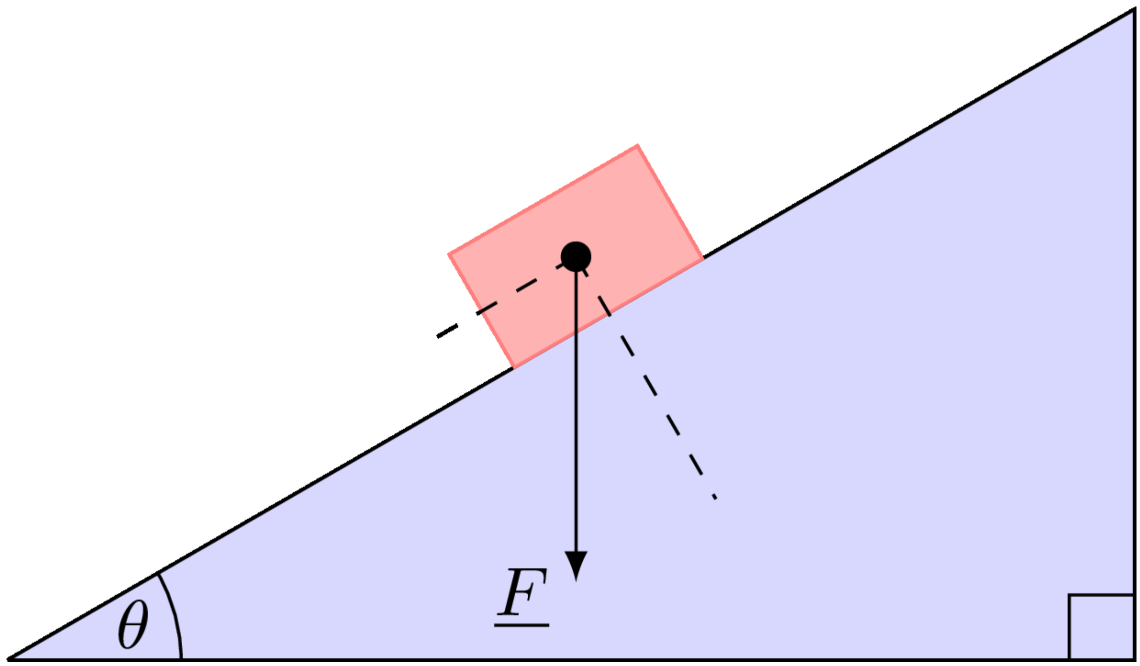

Figure 4.19: An object on a slope at angle θ, with a gravitational force labelled.

If we zoom into the object on the slope and label the angles, we can then use trigonometry to work out how much force is in each direction:

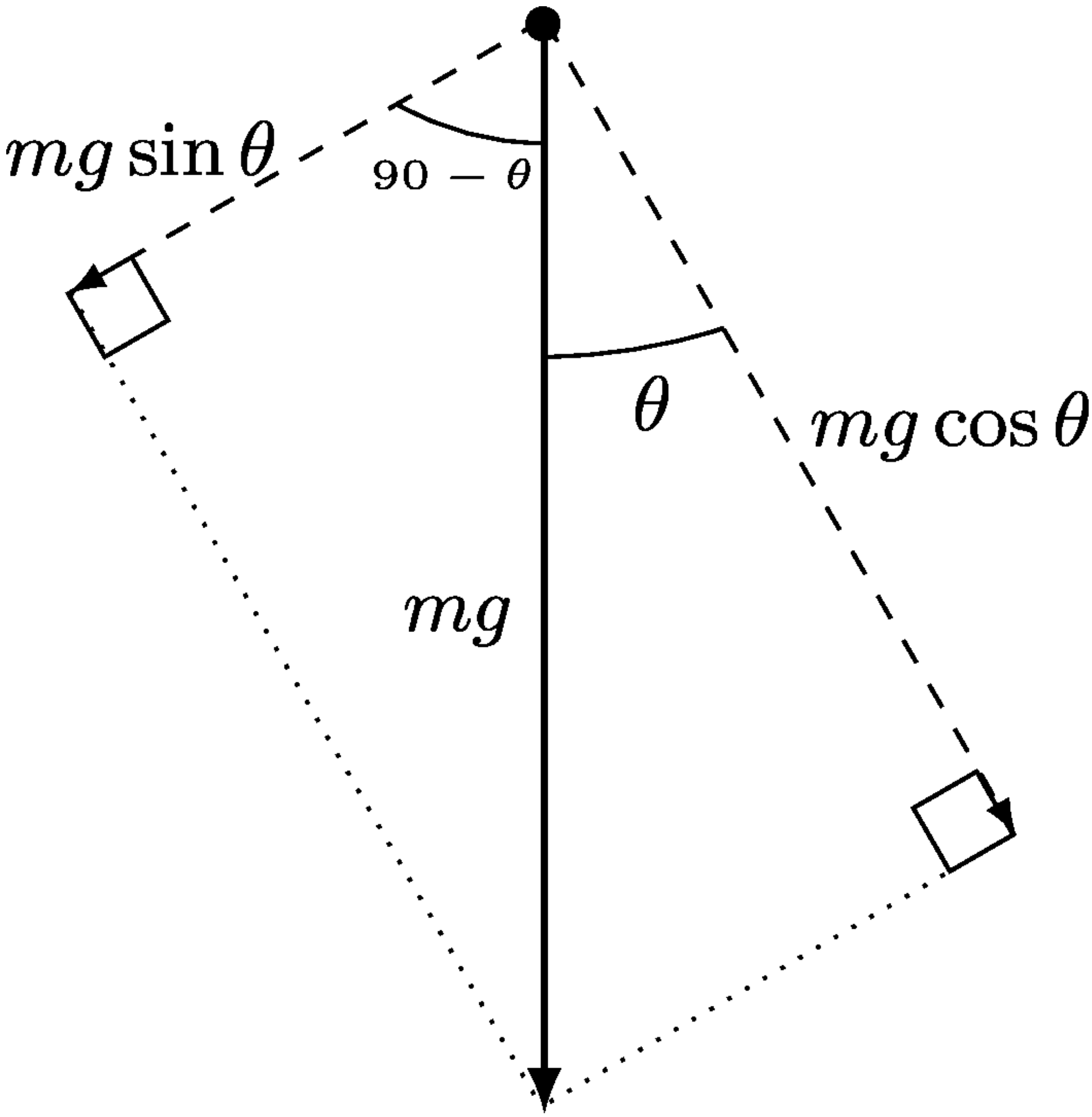

Figure 4.20: A zoom into the object on the slope with angles labelled.

Labelling these forces on the diagram we get the following version, showing how much of the force of mg Newtons is *resolved* in each direction:

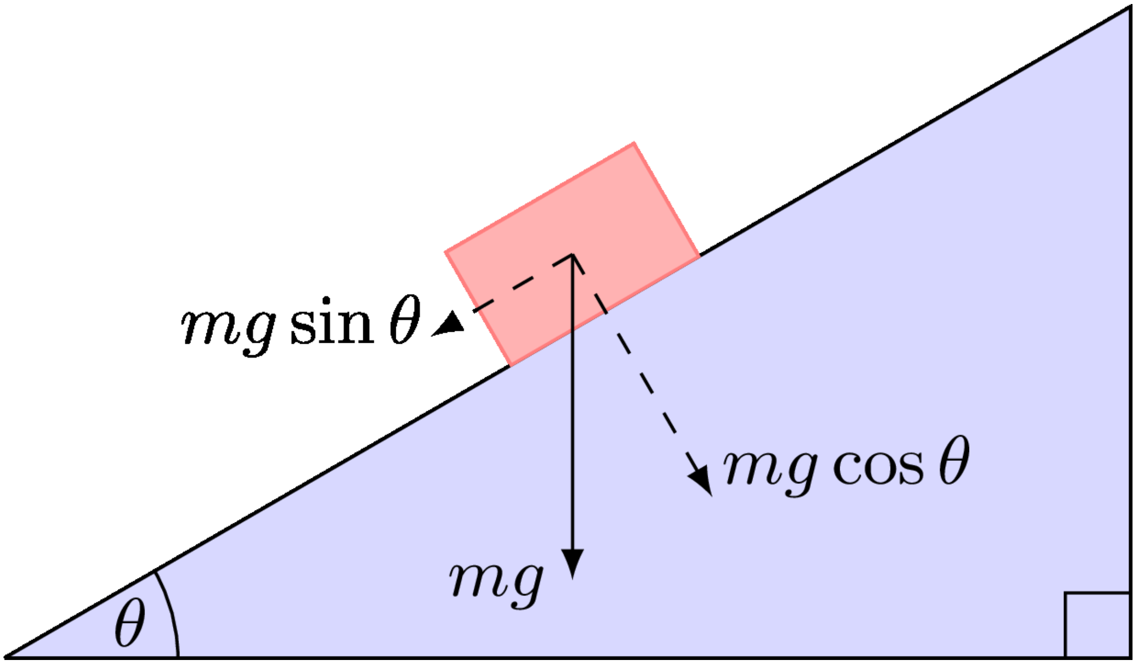

Figure 4.21: The object on the slope, with force size mg resolved into directions.

There is in fact a shortcut which skips using trigonometry, via our Scalar Product. **This is the approach we shall learn here**.

We define the Scalar **projection** of a force onto a line by taking a vector $\underline{v}$ of length 1 in the direction of the line and then calculating:

$$\underline{F} \cdot \underline{v}.$$

This *projection* of a force onto a line tells you how much of the force acts in this direction. (The same as *resolving* forces seen above.)

Definition of a unit vector
A vector which has length equal to 1 is known as a **unit** vector. We could have defined this earlier when defining length, but it wasn't necessary.
For every possible direction there is always exactly only vector of length 1, so there are as many unit vectors as there are directions, i.e. infinity!
In a number of applications of scalar products, and vectors in general, we need to take a starting vector and change (or more technically *scale*) its length[a] to length 1. To do this you simply need to divide a vector by its length, or multiply it by the scalar $\frac{1}{\text{length}}$.
e.g. given a vector length 12, multiplying the vector by $\frac{1}{12}$ will create a new vector of length 1, without changing the vector's direction.

[a]not its direction

To find how much of a force acts in a particular direction, we just find the Scalar Product of our force with a vector of length 1 (i.e. a *unit* vector) in the direction we care about.

Using the scalar product to resolve a force
If we consider exactly the same object on the slope from Figure 4.19, then we can find the size of the gravitational force acting down the slope by calculating the Scalar Product of the force, $\underline{F} = mg$, with a vector of length 1 that points down the slope.

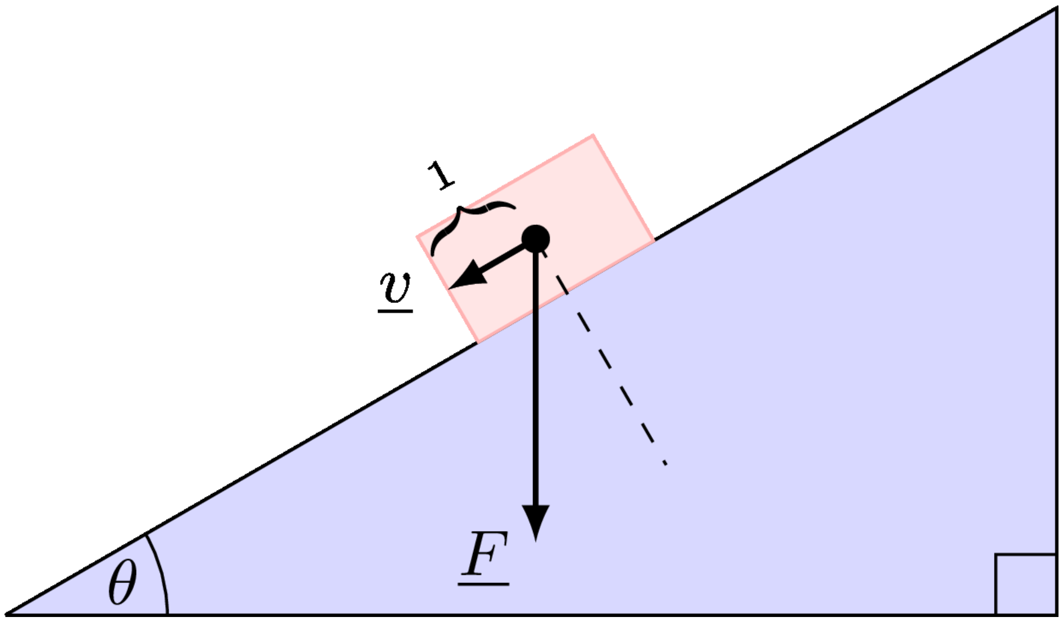

Figure 4.22: The required vector $\underline{v}$ using the Scalar Product approach.

You just need the vector $\underline{v}$, with length($\underline{v}$) = 1, as per Figure 4.22, and then the force's resolution/projection down the slope is calculated via

$$\underline{F} \cdot \underline{v}.$$

Note, in this example we haven't calculated $\underline{v}$ but in many examples you will already have a vector from previous calculations you can use for $\underline{v}$ – which you perhaps need to scale first to make it of length 1.

Numerical example of Scalar projection

Consider a force $\underline{F} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$, which you can check has magnitude[a] of 5 and so represents a force of 5 Newtons, and you wish to know how much effect this force has in the Southeast direction...
We start with a vector which points in the Southeast direction like

$$\underline{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Unfortunately, this vector has length

$$\sqrt{(1)^2 + (-1)^2} = \sqrt{2} \neq 1$$

So we divide $\underline{u}$ by its length to create a new vector which points in the same direction, but is of length 1[b]. So we divide $\underline{u}$ by $\sqrt{2}$ via scalar multiplication:

$$\text{Let } \underline{v} = \frac{1}{\sqrt{2}} \times \underline{u} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \approx \begin{pmatrix} 0.707 \\ -0.707 \end{pmatrix}$$

This vector $\underline{v}$ points in the Southeast direction and had length 1 so the size of the force $\underline{F}$ which acts in the Southeast direction is calculated as

$$\underline{F} \cdot \underline{v} \approx \begin{pmatrix} 4 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 0.707 \\ -0.707 \end{pmatrix} = 0.707 \text{ N}.$$

Notice that the original force $\underline{F}$ actually had strength of 5 N, but not much of it acts in the Southeast direction, only 0.707 Newtons worth.

[a] length
[b] think about it!

4.11.4 Calculating the "work done" by a force

Another important concept in mechanics is the *work done* by a force. This quantifies how much energy is transferred by the force during the movement of an object.

It has a formal definition, which is actually very simple.

The definition of work done
If a force $\underline{F}$ acts upon an object, and results in a displacement vector (from any original position) represented by $\underline{d}$ then

$$\text{Work done } = \underline{F} \cdot \underline{d}.$$

This says that the *Work done* is just the Scalar Product of the force acting with the displacement which occurred.

If a number of different forces, $\underline{F}_1, \underline{F}_2, \underline{F}_3, ...$ all act upon an object you can work out individually how much work each force contributed to an overall displacement by separately calculating $\underline{F}_1 \cdot \underline{d}, \underline{F}_2 \cdot \underline{d}, \underline{F}_3 \cdot \underline{d}, ...$ The overall work done is then the sum of these individual answers.

Example
If the wind, represented by a force $\underline{F}$, moves an object from a location P to a location Q then how much work is done by the wind?
We start by calculating the displacement, $\overrightarrow{PQ}$, see Section 4.8.
Then we calculate

$$\text{Work done } = \underline{F} \cdot \overrightarrow{PQ}$$

For example, if $\underline{F} = \begin{pmatrix} 25 \\ 5 \end{pmatrix}$ and the object is moved from point P with co-ordinates $(1, 5)$ to point Q with co-ordinates $(10, 18)$ then the work done is

$$\begin{pmatrix} 25 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 9 \\ 13 \end{pmatrix} = 25 \times 9 + 5 \times 13 = 290 \text{ (Joules)}.$$

Practice

How much work is done by the force $\underline{F} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$, in moving an object from location $P = (3, 1)$, to location $Q = (-4, 3)$?
Answer[a]

[a] 20 Joules

4.12 Vectors in more than 2 dimensions

Two-dimensional (2D) vectors are easy to visualize and also easy to sketch as they can be drawn on a flat piece of paper. Three dimensional vectors, however, require more spatial reasoning skills to visualize, and are harder to draw on 2D surfaces like computer screens.

Here's an attempt:

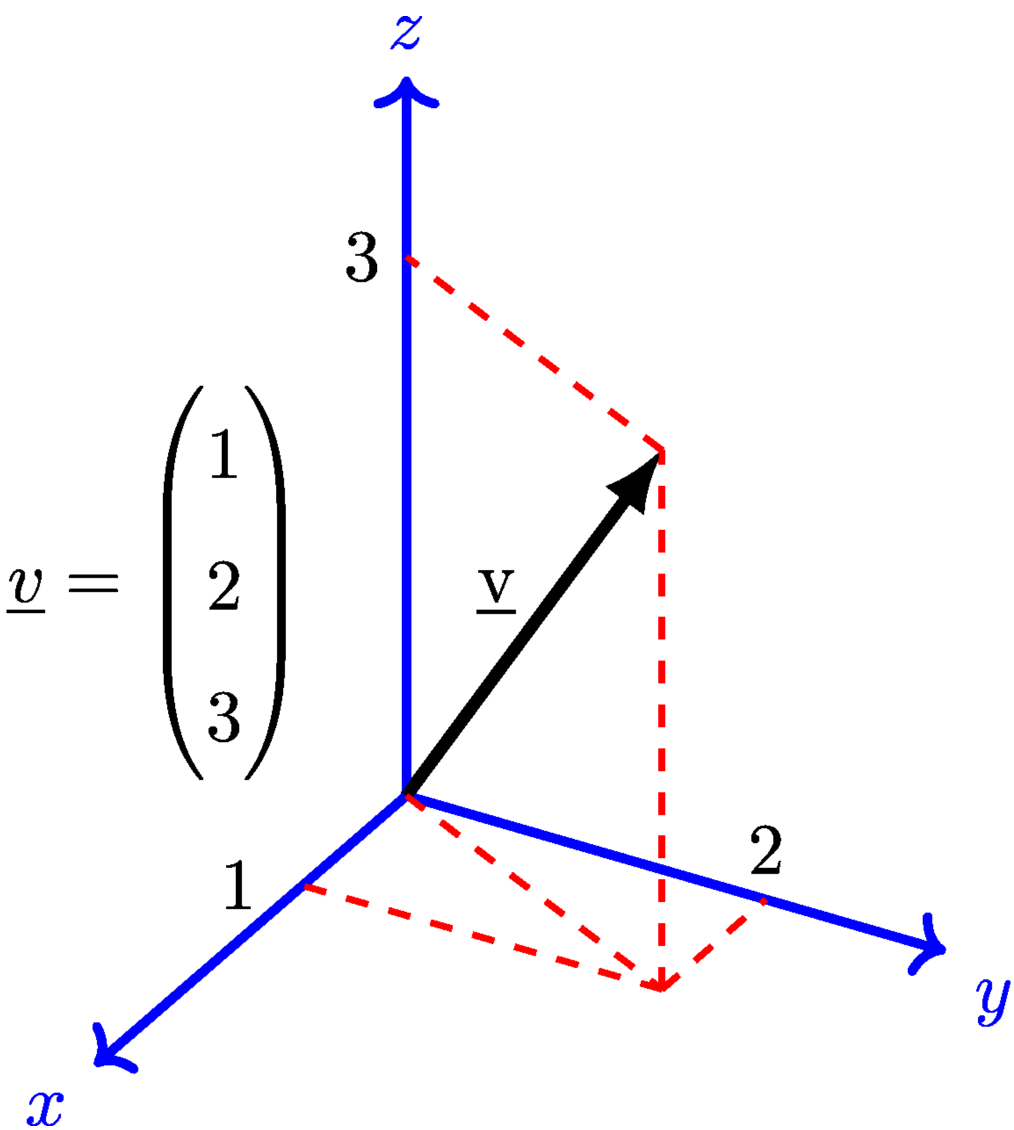

Figure 4.23: An attempt to show a 3D-vector (on a 2D screen).

As you can see the standard rectangular notation matrix format has been extended to just contain a longer column vector like this:

$$\underline{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Indeed one can generalize this process to as many dimensions as you wish, writing longer and longer vectors, however our human brains aren't very good at visualizing beyond three dimensions, and there aren't as many real-world applications to vectors beyond 3-dimensions. We read the first component as the x-coordinate, then the second component as the y-coordinate and finally the third component as the z-coordinate.

In the $\underline{i}, \underline{j}$ notation introduced in Section 4.9, you can just introduce $\underline{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ to represent the third dimension.

Extending the existing $\underline{i}$ and $\underline{j}$ in the natural way as $\underline{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\underline{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. This vector $\underline{v}$ can then be written $\underline{v} = \underline{i} + 2\underline{j} + 3\underline{k}$.

4.12.1 Adding, subtracting and scaling 3D vectors

The great news is that the rules for performing algebra on vectors in 2D extend completely as expected to 3D vectors.

- Addition and subtraction work the same. You just have a third calculation to do each time. (See equations (4.2) and (4.3))
- Scalar multiplication is the same. You just have to multiple all **three** components by the scalar. (See equation (4.4))

Examples

$$\begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1+2 \\ 3+(-1) \\ (-4)+3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$$

and

$$3 \times \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -6 \end{pmatrix}$$

4.12.2 Lengths for 3D vectors

Amazingly even Pythagoras can be extended to work in more than two dimensions. The standard formula is that you need to calculate the sum of the squares of **all** the coordinates.

So if $\underline{v} = \begin{pmatrix} -2 \\ 7 \\ 3 \end{pmatrix}$ then

$$\text{length}(\underline{v}) = |\underline{v}| = \sqrt{(-2)^2 + 7^2 + 3^2} = \sqrt{62}.$$

4.12.3 Scalar Product for 3D vectors

Even our Scalar Product formula can be extended quite easily to three dimensions. There's just a third product to add on the end of the calculation, as an extension to our 2D version earlier.

If $\underline{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\underline{w} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ then

$$\underline{v} \cdot \underline{w} = xa + yb + zc.$$

An amazing fact is that the second formula for the Scalar Product (equation (4.5)), i.e.

$$\underline{v} \cdot \underline{w} = \text{length}(\underline{v}) \times \text{length}(\underline{w}) \times \cos(\theta)$$

is totally unchanged! The angle between the two vectors is now the angle in 3D space between the two vectors. This means the widest angle visible between them, visible to a viewer who moves around so that both vectors are perpendicular to their vision.

The following properties and facts about Scalar Products still work in 3 dimensions:

- The *orthogonality* rule, in Section 4.11.2, so a Scalar Product of zero still means two vectors form a right-angle.
- The formula for *work done* remains the same,
- The formula for *scalar projection onto another vector* remains the same too.

All these properties still working is why this definition of the (Scalar) Product between two vectors is so useful and powerful in many areas.

Here's an example of a Scalar Product calculation for two 3-dimensional vectors:

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 11 \\ 4 \\ 7 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 11 \\ 4 \\ 7 \end{pmatrix} = 1 \times 11 + \ldots$$

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 11 \\ 4 \\ 7 \end{pmatrix} = 1 \times 11 + 2 \times 4 + \ldots$$

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 11 \\ 4 \\ 7 \end{pmatrix} = 1 \times 11 + 2 \times 4 + 3 \times 7 = 40$$

Figure 4.24: Step-by-step Scalar Product calculation of two 3D-vectors.

Let us see how the 3-dimensional Scalar Product can be used to find angles.

Example
A triangle in 3D has vertices[a] at $A = (3, 2, 4)$, $B = (1, -4, 5)$ and $C = (2, 0, -1)$. Determine the angle between the sides AB and AC.
The angle between the sides AB and AC will be the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.
So we begin by finding

$$\overrightarrow{AB} = \begin{pmatrix} 1-3 \\ -4-2 \\ 5-4 \end{pmatrix} = \begin{pmatrix} -2 \\ -6 \\ 1 \end{pmatrix},$$

and

$$\overrightarrow{AC} = \begin{pmatrix} 2-3 \\ 0-2 \\ -1-4 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -5 \end{pmatrix}.$$

To calculate an angle between two vectors we use the Scalar Product formula ((4.5)):

$$\underline{v} \cdot \underline{w} = \text{length}(\underline{v}) \times \text{length}(\underline{w}) \times \cos(\theta)$$

In our case, we can easily calculate

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = 2 + 12 - 5 = 9,$$

so re-arranging the Scalar Product formula for the angle θ we get

$$\cos(\theta) = \frac{9}{\text{length}(AB)\,\text{length}(AC)}.$$

So next we calculate the lengths of our sides, by using (3-dimensional) Pythagoras,

$$\text{length}(AB) = \sqrt{(-2)^2 + (-6)^2 + 1^2} = \sqrt{41}, \text{ and}$$
$$\text{length}(AC) = \sqrt{(-1)^2 + (-2)^2 + (-5)^2} = \sqrt{30}.$$

So, $\cos(\theta) = \frac{9}{\sqrt{41}\sqrt{30}} = 0.2566\ldots$. Finally, using $\cos^{-1}$ we obtain

$$\theta = \cos^{-1}(0.2566\ldots) = 75.13^\circ.$$

[a] a technical term for corners!

4.12.4 Polars and other formats for 3D vectors

This is the only topic where 3D vectors become a little messier. You can no longer describe a vector in 3D with just a length and a single angle. Indeed there are competing ways to describe vectors in 3D using at least one angle. We shall not cover them here since they don't get enormous usage outside of certain specific real-world applications, but an interested reader could go and look them up. The two main formats are:

- Spherical (polar) format, and
- Cylindrical (polar) format.

You will have come across at least one of these before, because the Longitude and Latitude system of co-ordinates for the surface of Planet Earth are actually just *Spherical polar co-ordinates* where the length is fixed at the Earth's radius.

4.13 The Vector Product of two vectors

As promised we shall briefly introduce the other invented method for multiplying vectors. This second type of multiplication is called the Vector Product (or the Cross product), it is written with a $\times$ symbol, e.g. $\underline{v} \times \underline{w}$.

First two warnings...

The Vector Product for vectors is only designed for 3-dimensional vectors.

The Vector Product is messier to calculate than the Scalar Product.

As with the Scalar Product, the name tells you what type of object you get as a result. Let's see a definition of the Vector Product.

The Vector Product (also called Cross Product) of two vectors, $\underline{v}$ and $\underline{w}$ is a calculation written

$$\underline{v} \times \underline{w}.$$

It is **only defined** if both $\underline{v}$ and $\underline{w}$ are 3-dimensional, e.g.

$$\underline{v} = \begin{pmatrix} 4 \\ 1 \\ -5 \end{pmatrix} \text{ and } \underline{w} = \begin{pmatrix} 2 \\ -9 \\ -3 \end{pmatrix}$$

The general formula is not pretty, and looks like this:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} bf - ce \\ cd - af \\ ae - bd \end{pmatrix} \tag{4.7}$$

Note that unlike the Scalar Product (where the output is a number/scalar), the output of the Vector Product is a vector!

It's not recommended to try and remember this formula, instead there are a number of different ways to relate this formula to other topics in maths which make it more memorable. Probably the easiest method is via what is known as the **determinant** of a matrix, which we shall cover in Chapter 5, specifically in Section 5.3.8 and Equation (5.5).

For now we shall just see an example of the calculation, and then make some remarks upon where it is useful.

Taking the sample vectors from above, i.e.

$$\underline{v} = \begin{pmatrix} 4 \\ 1 \\ -5 \end{pmatrix} \text{ and } \underline{w} = \begin{pmatrix} 2 \\ -9 \\ -3 \end{pmatrix} \tag{4.8}$$

We can calculate the Vector Product via formula (4.7).

$$\begin{aligned} \begin{pmatrix} 4 \\ 1 \\ -5 \end{pmatrix} \times \begin{pmatrix} 2 \\ -9 \\ -3 \end{pmatrix} &= \begin{pmatrix} (1)(-3) - (-5)(-9) \\ (2)(-5) - (4)(-3) \\ (4)(-9) - (1)(2) \end{pmatrix} \\ &= \begin{pmatrix} (-3) - (45) \\ (-10) - (-12) \\ (-36) - (2) \end{pmatrix} \\ &= \begin{pmatrix} -48 \\ 2 \\ -38 \end{pmatrix} \end{aligned}$$

Definitely not a pretty calculation, plenty of scope for errors, especially with numerous negative signs.

4.14 Applications of the Vector Product

4.14.1 Finding a "normal" to 2 vectors

The word *normal* in the context of vectors is an adjective meaning *perpendicular to*, *orthogonal to*, or "makes a right-angle with".

One amazing property[19] of the Vector Product is that

$$\underline{v} \times \underline{w} \text{ is always perpendicular to both } \underline{v} \text{ and } \underline{w}.$$

So to find a vector which is *normal* to two other vectors you can just calculate their Vector Product. If you think about it geometrically, in 3-dimensions the answer to this question is unique excepting scalar multiples of the answer. That is, if $\underline{n}$ is normal to $\underline{v}$ and $\underline{w}$ then the only other vectors which are also normal to $\underline{v}$ and $\underline{w}$ are vectors that look like $k\underline{n}$ for all constant values of k.

Let us check this orthogonality claim for $\underline{v} \times \underline{w}$ for an example, by using our orthogonal property from the Scalar Product we saw in Section 4.11.2.

[19] and probably the most useful property

We can check this with our example we used above to calculate a Vector Product. We begin with

$$\underline{n} = \begin{pmatrix} -48 \\ 2 \\ -38 \end{pmatrix}$$

and let us first calculate $\underline{n} \cdot \underline{v}$ where $\underline{v}$ and $\underline{w}$ were defined as

$$\underline{v} = \begin{pmatrix} 4 \\ 1 \\ -5 \end{pmatrix} \text{ and } \underline{w} = \begin{pmatrix} 2 \\ -9 \\ -3 \end{pmatrix}.$$

$$\begin{aligned} \underline{n} \cdot \underline{v} &= \begin{pmatrix} -48 \\ 2 \\ -38 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 1 \\ -5 \end{pmatrix} \\ &= -48 \times 4 + 2 \times 1 + (-38) \times (-5) \\ &= -192 + 2 + 190 = 0 \quad \checkmark \end{aligned}$$

The Scalar Product was zero, so the two vectors are indeed perpendicular[a] in 3D.
The second calculation, verifying that $\underline{n} \cdot \underline{w} = 0$ is left for you as an exercise.

[a] or orthogonal, or at-right-angles

4.14.2 Calculating the moment of a force

The *moment of a force*, also called a *torque*, is a physics concept representing the turning force around some location. Since these notes are focussed upon the mathematical elements we shall just provide a definition here and not go into the physics.

Definition of moment
If we have a force, $\underline{F}$, which acts through a point in space we call B, then the *moment of this force about another point* A is defined as

$$\overrightarrow{AB} \times \underline{F}.$$

Recall the notation $\overrightarrow{AB}$ means the vector joining the point A to the point B, see Section 4.8.

4.14.3 Other uses of the Vector Product

There are a few other more obscure uses of the Vector Product, they are not worth discussing here, but an interested reader could go and look up *shearing force* and the *volume of a parallelepiped*!

4.15 Further practice exercises

There are many more vectors examples for you try, especially for these latter topics, in the specific further exercises Section 7.

As with the algebra chapter, reinforcing your learning is best done by trying examples, so you are recommended to seek out your own further exercises too.

4.16 Summary of chapter

Having worked through this chapter and attempted the exercises you should now have developed understanding of the following topics:

- know what vectors (and scalars) are and identify common notations for them;
- use Cartesian/Rectangular or Polar notations for vectors and be able to convert between them;
- perform basic algebra like addition and scalar multiplication of vectors;
- use vectors to represent relative positions of objects;
- define, calculate and use the Scalar Product of two vectors in various applications;
- define, calculate and use the Vector Product of two three-dimensional vectors in various applications.

Chapter 5

Introduction to Matrices

A matrix is just a collection of numbers arranged into a grid. They are always drawn with brackets[1] around them. Here are a few examples of matrices[2] of different shapes and sizes,

$$\begin{pmatrix} 4 & 0 & 3 & -1 \end{pmatrix}, \begin{pmatrix} 1 & \sqrt{2} \\ y & 7 \end{pmatrix}, \begin{pmatrix} 3 & 9 \\ -1 & 7 \\ 0 & 3.4 \end{pmatrix}, \begin{pmatrix} 4 & 5 & 5 \\ 1 & -2 & -9 \\ 1 & 0 & 0 \end{pmatrix}. \tag{5.1}$$

In this introduction most of the numbers in the matrix grid will be integers, to make your calculations easier, but they don't need to be. As you can see above you can even just insert algebra letters, to represent unknowns.

Matrices have a very wide range of practical uses, here are just a few you may come across:

- For representing movements and performing calculations in computer graphics;
- For representing mechanical systems and movements;
- For representing electrical circuits and calculating currents and voltages;
- For storing, then manipulating statistical data;
- As a large family of objects useful in cryptography;
- For representing data in, and then solving, business optimization problems.

The two data applications above both hint at the use of spreadsheet software, which naturally contains boxes in a grid formation for storing numbers.

We have actually seen matrices before, when we discussed the Rectangular/Cartesian format for vectors. In 4.3 we introduced a notation which placed the components of a vector into a column inside parentheses as follows:

$$\underline{v} = \begin{pmatrix} 3 \\ -3 \\ 7 \end{pmatrix}.$$

This format is just a special case of a matrix with grid of width 1 and height 3.

Fundamentally matrices are just a clever notation for holding many numbers in a grid format at the same time. However, as with lots of notation in maths, once we have a nice notation if we invent nice ways to manipulate it and do algebra then we can later make solving very complex problems relatively easy. We shall

[1] or parentheses
[2] the plural of matrix

see exactly this for solving systems of linear equations later (i.e. solving simultaneous equations in Sections 5.6, 5.7 and 5.8).

5.1 Terminology

In absolute consistency with the notation used with vectors, the numbers in a matrix can be described by which row and which column they lie in. In keeping with terminology for vectors, rows are horizontal and columns are vertical.

A matrix can therefore be thought of as a collection of rows stacked on top of each other, or as a collection of columns placed side-by-side.

5.1.1 Describing the shape of a matrix

When it comes to describing the shape we just need to tell the reader how many rows and how many columns the matrix has. Note that matrices are always a full grid, every row is the same length and every column is the same length (height).

Formal shape definition
A matrix with 3 rows and 2 columns is called a 3-by-2 matrix, or a (3×2) matrix.
In general, if it has m rows and n columns then it is called an m-by-n matrix.
Notice that the order of m and n will matter[a] so it's important to agree which number comes first, and which second.
The same memory aids exist here remembering the usage of rows and columns, from when we discussed vectors.
fi**R**st = **R**ows
se**C**ond = **C**olumns
Additionally, for anyone who confuses the meanings of rows and columns... seats in cinemas are arranged in rows, and good old Roman columns on buildings stand vertically.

[a] if they aren't the same number

This m-by-n notation will be important if we use a letter like M in algebra to describe a matrix, because you cannot tell from just the letter M what shape it is. Naturally if you have written out the matrix in full inside brackets then it will be obvious what shape it is!

5.1.2 Describing elements of a matrix

If we wish to refer to a specific number inside a matrix, we can identify it by which intersection of row and column it lies at.

For example, in this large matrix,

$$M = \begin{pmatrix} 3 & 1 & 8 & -1 \\ 2 & 3 & \color{red}{0} & 9 \\ 3 & -7 & 5 & 9 \end{pmatrix}$$

the element at row 2, column 3 is a 0 (zero)[a].
In repeated consistency with the previous section, we can shorten the description of the element in Row 2 and in Column 3, and just say that the $(2, 3)$ element of the matrix is 0.

[a] it has also been highlighted in red

Various authors of books may also describe *elements* of a matrix as *entries*. Additionally the $(2, 3)$ element of M can also be called $M(2, 3)$ or $M_{2,3}$ or even m_{23}. This last notation without a comma is only safe if the number of rows and columns is small, otherwise you might not be able to tell where the two numbers separate.

5.1.3 Comparing matrices

In most work with matrices we generally only deal with one (or two) different shapes within a single question. Indeed a lot of the most interesting uses of matrices are specific to what we shall later call *square matrices* (which have the same number of rows as they have columns).

When it comes to directly comparing two matrices, we only say that two matrices are *equal* if they are exactly the same shape and all the elements match exactly.

Before we conclude this terminology section, a few exercises to provide you with some practice with the new terms.

```
#> [Embedded question appears here in html version
#>  the hyperlink is provided below]
```

Link to Numbas on the web

Question 1: For each of the matrices in the original list in this chapter (see (5.1)) identify what shape they are in the m-by-n notation.
Question 2: Were any pair of the matrices in those examples the same shape as each other?
Question 3: Finally, what shape is this matrix and what is its $(1,4)$ element?

$$\begin{pmatrix} 4 & -7 & 9 & 8 \\ 0 & 91 & -3 & -1 \\ 2 & 5 & 7 & 2 \\ 1 & 2 & 16 & -3 \end{pmatrix}$$

Answers:
Question 1[a]
Question 2[b]
Question 3[c]

[a]The shapes of the matrices are... 1-by-4, 2-by-2, 3-by-2 and 3-by-3.
[b]None were of the same shape as each other.
[c]Finally the $(1,4)$ element of this 4-by-4 matrix is 8. We needed Row 1, Column 4 for the $(1,4)$ element.

5.2 Algebra operations

Whenever we introduce a new mathematical object, such as a matrix, we need to also check whether the standard algebra we do with numbers can be applied to it too. Sometimes new mathematical objects require special meanings for addition, multiplication, division, etc... However, the good news is that for matrices the basic rules are very similar. The main complication occurs in the same place as it did with vectors. Namely, that when we think about multiplication there are two choices: do we want to multiply by a scalar or by another matrix? We shall see the answers later.

As with all new objects the idea is to define how to do algebra in the most sensible way so that we can continue to perform as much of our normal algebra as possible.

5.2.1 Adding and subtracting matrices

These calculations are going to be nice and straightforward.

First a couple of warnings...

Some things you cannot do:

- You cannot add a single number to a matrix so $4 + \begin{pmatrix} 2 & -3 \\ 1 & 5 \end{pmatrix}$ doesn't make sense.
- You cannot add two matrices if they are not exactly the same shape.

Now for the good news, once *you have two matrices of identical shape*, adding them together is as easy as just separately adding every element with its matching position element in the other matrix. Subtraction is the same, you just subtract the second matrix elements from the first, one-by-one. Let's see one example of each, after which is should be very obvious how it works.

To add two matrices they need to be the same shape, and then you add together corresponding location elements like this:

$$\begin{aligned}
\begin{pmatrix} 1 & 4 \\ -1 & 0 \\ 3 & 1 \\ 4 & -1 \end{pmatrix} + \begin{pmatrix} 4 & 2 \\ 1 & -4 \\ 9 & -5 \\ 1 & 0 \end{pmatrix} &= \begin{pmatrix} 1+4 & 4+2 \\ -1+1 & 0+-4 \\ 3+9 & 1+-5 \\ 4+1 & -1+0 \end{pmatrix} \\
&= \begin{pmatrix} 5 & 6 \\ 0 & -4 \\ 12 & -4 \\ 5 & -1 \end{pmatrix}
\end{aligned}$$

Eight separate additions were performed, check them!
To subtract one matrix from another it's the same idea. They need to be the same shape, and you just need to make sure you know which number is on the left and which on the right:

$$\begin{aligned}
\begin{pmatrix} 7 & -4 \\ 2 & 4 \end{pmatrix} - \begin{pmatrix} 1 & -3 \\ -5 & 6 \end{pmatrix} &= \begin{pmatrix} 7-1 & -4--3 \\ 2--5 & 4-6 \end{pmatrix} \\
&= \begin{pmatrix} 6 & -1 \\ 7 & -2 \end{pmatrix}
\end{aligned}$$

Here four separate standard subtractions were performed, check them!

5.2.2 Scalar multiplication of matrices

Also relatively easy is the idea of multiplying a whole matrix by a number[3]. For this calculation you just multiply every element of the matrix by this number. If M is a matrix, you can then talk about $2M$, and $3M$ and kM for any constant k.

Let's see an example combining addition and multiplication by a scalar.

[3] a scalar

Let $M = \begin{pmatrix} 1 & 4 \\ 0 & 3 \end{pmatrix}$ and $N = \begin{pmatrix} -2 & 3 \\ 1 & -5 \end{pmatrix}$ both be 2-by-2 matrices.
First let's calculate $3M$:

$$3M = 3\begin{pmatrix} 1 & 4 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 12 \\ 0 & 9 \end{pmatrix}$$

You should check each element matches what you were expecting.
Now for a linear combination of M and N, let's calculate $3M - 2N$:

$$\begin{aligned} 3M - 2N = \begin{pmatrix} 3 & 12 \\ 0 & 9 \end{pmatrix} - 2\begin{pmatrix} -2 & 3 \\ 1 & -5 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 12 \\ 0 & 9 \end{pmatrix} - \begin{pmatrix} -4 & 6 \\ 2 & -10 \end{pmatrix} \\ &= \begin{pmatrix} 7 & 6 \\ -2 & 19 \end{pmatrix} \end{aligned}$$

Again, check the calculations here to convince yourself you know what's happening.

```
#> [Embedded question appears here in html version
#>  the hyperlink is provided below]
```

Link to Numbas on the web

5.2.3 Matrix multiplication using the Scalar product

As forewarned, multiplying a matrix by another matrix (known as *matrix multiplication*) is the most complicated part of working with matrices. We want a way of multiplying matrices by each other which has practical applications and is useful. The good news is that there is indeed a way to do this which is extremely useful in many applications.

The bad news is that the process involves a lot more calculations than the other algebra seen so far. None of the calculations are complex, but there are quite a lot of them. Conveniently if you have already studied the Scalar Product[4] of two vectors (see 4.10) then the method can be understood more easily.

So first we shall remind ourselves how the Scalar Product of two vectors of the same shape works. Remember first that a vector is just a matrix with 1 column.

[4] also called the *Dot Product*

Scalar Product definition (reminder)
If we have two vectors of the same shape (i.e. they have the same number of elements) then their Scalar Product is calculated by

- Multiplying each element of one vector by the element of the other vector in the corresponding position, then
- Adding together all these answers.

Here's the example we saw before,

For vectors $\underline{v} = \begin{pmatrix} 8 \\ -2 \end{pmatrix}$ and $\underline{w} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ their Scalar Product is

$$\underline{v} \cdot \underline{w} = \begin{pmatrix} 8 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 8 \times 3 + (-2) \times 4 = 16.$$

As an example for 3-dimensional vectors, $\underline{v} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$ and $\underline{w} = \begin{pmatrix} 5 \\ 8 \\ -1 \end{pmatrix}$ their Scalar Product is

$$\begin{aligned} \underline{v} \cdot \underline{w} &= \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 8 \\ -1 \end{pmatrix}, \\ &= 2 \times 5 + (-1) \times 8 + 4 \times (-1), \\ &= 10 - 8 - 4 = -2. \end{aligned}$$

It was worth seeing this Scalar Product definition (again) because our method for finding the product of two matrices is actually just going to be a collection of different Scalar Products! In words, the product of two matrices (called A and B), is going to be a new matrix whose elements are the separate Scalar Products created by matching every row from A with every column from B. First a couple of warnings...

Warning 1:
When multiplying matrices together the *order* matters, so $A \times B$ and $B \times A$ are different calculations[a]
Warning 2:
Two matrices A and B can only be multiplied together to create AB if ...
"The number of columns of A" equals "The number of rows of B".

[a] We almost always omit the $\times$ symbol between matrices. And just write the one-letter names in order, so that $A \times B$ is written AB and $B \times A$ is written BA. But this is all just notational convenience.

Example
We consider the product of two matrices, called A and B, which are defined as follows

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 11 & 2 \\ 4 & 5 \\ 7 & 8 \end{pmatrix}$$

The product

$$AB$$

is created by writing A to the left of B[a]

$$AB = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 11 & 2 \\ 4 & 5 \\ 7 & 8 \end{pmatrix}$$

Notice that the **length of each row of** A[b] matches the **height of each column of** B[c]. Both of these quantities are equal to 3.

[a] Warning: this will be different to putting B to the left of A
[b] which is also called its number of columns!
[c] also called its number of rows!

This is how we start to write the product of two matrices A and B. But how about the method for actually calculating the product? Well in words, we have to do the following...

To calculate the product of two matrices involves lots of separate calculations:

- One calculation for **every combination** using... a **row from** A and a **column from** B!
- In our example above A has 3 *rows*, so there are 3 row choices.
- In our example above B has 2 *columns*, so there are 2 column choices.
- **Together** this means there are $3 \times 2 = 6$ combinations of row and column choices. For completeness we shall list them all here:

1. Row 1 (of A) with Column 1 (of B)
2. Row 1 (of A) with Column 2 (of B)
3. Row 2 (of A) with Column 1 (of B)
4. Row 2 (of A) with Column 2 (of B)
5. Row 3 (of A) with Column 1 (of B)
6. Row 3 (of A) with Column 2 (of B)

You will need to *systematically go through all 6 combinations* to calculate the product AB. You are creating the entries for AB one at a time.

For every (row from A,column from B)-pair you need to **calculate the Scalar Product**. You insert the answer in the corresponding position in the answer matrix. i.e. if you used Row i (of A) and Column j (of B) then their Scalar Product should be inserted in Row i, Column j of the product AB.

We'll write out the formal definition in words, and set an exercise for you to try, then do a fully worked example. The example will likely make more sense than the wordy definition when you first read it.

Definition of matrix multiplication
Once you have selected a row from A and a column from B you treat them both as vectors of the same length and **find their Scalar Product**.
The answer to multiplying two matrices together is another matrix, and the answer has shape[a] which matches the *number of rows of* A and the *number of columns of* B. In this case A had *3 rows* and B had *2 columns*, so AB has *3 rows* and *2 columns*. This will become much clearer when we learn what to do with the Scalar Products explained above.
For **each row of** A and for **each column of** B you find the Scalar Product of those vectors and put the answer in the location described by where you got the vectors from!
So, using Row 3 of A and Column 1 of B means you calculate this Scalar Product

$$\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \cdot \begin{pmatrix} 11 \\ 4 \\ 7 \end{pmatrix} = 77 + 32 + 63 = 172$$

and the answer of 172 is placed in Row 3, Column 1 of the answer, i.e. in AB it appears like this

$$AB = \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \\ 172 & \cdot \end{pmatrix}.$$

Note: We rotated Row 3 of A so we could write it next to Column 1 of B to do the Scalar Product.

[a] how many rows and columns it has

Practice
Using the method outlined above try and calculate the five missing values in AB (represented by dots) here:

$$AB = \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \\ 172 & \cdot \end{pmatrix}$$

As a reminder here are A and B in this example:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \quad B = \begin{pmatrix} 11 & 2 \\ 4 & 5 \\ 7 & 8 \end{pmatrix}$$

Hint: to find the Row 1, Column 2 entry of AB, you need to find the Scalar Product of Row 1 from A with Column 2 from B. [a]

[a] Remember it's always **R**ows fi**R**st and **C**olumns se**Co**nd.

It's **strongly recommended** that you watch this video to see matrix multiplication in action:

```
#> [Video appears here in html version
#>  the hyperlink is provided below]
#> [This pdf version instead contains detailed text steps below]
```

Alternative direct video link: https://gcu.planetestream.com/View.aspx?id=5555~4u~vB7eUMN3

Now let's do the full worked calculation of AB, slowly.
From *Row 1* of A and *Column 1* of B you calculate this Scalar Product

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 11 \\ 4 \\ 7 \end{pmatrix} = 1 \times 11 + 2 \times 4 + 3 \times 7 = 40$$

and the answer of 40 is placed in *Row 1*, *Column 1* of the answer.

From *Row 1* of A and *Column 2* of B you calculate this Scalar Product

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} = 1 \times 2 + 2 \times 5 + 3 \times 8 = 36$$

and the answer of 36 is placed in *Row 1*, *Column 2* of the answer.

From *Row 2* of A and *Column 1* of B you calculate this Scalar Product

$$\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 11 \\ 4 \\ 7 \end{pmatrix} = 4 \times 11 + 5 \times 4 + 6 \times 7 = 106$$

and the answer of 106 is placed in *Row 2*, *Column 1* of the answer.

From *Row 2* of A and *Column 2* of B you calculate this Scalar Product

$$\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} = 4 \times 2 + 5 \times 5 + 6 \times 8 = 81$$

and the answer of 81 is placed in *Row 2*, *Column 2* of the answer.

From *Row 3* of A and *Column 1* of B you calculate this Scalar Product

$$\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \cdot \begin{pmatrix} 11 \\ 4 \\ 7 \end{pmatrix} = 7 \times 11 + 8 \times 4 + 9 \times 7 = 172$$

and the answer of 172 is placed in *Row 3*, *Column 1* of the answer.

From *Row 3* of A and *Column 2* of B you calculate this Scalar Product

$$\begin{pmatrix}7\\8\\9\end{pmatrix} \cdot \begin{pmatrix}2\\5\\8\end{pmatrix} = 7\times 2 + 8\times 5 + 9\times 8 = 126$$

and the answer of 126 is placed in *Row 3*, *Column 2* of the answer.

So our final answer is:

$$\begin{aligned} AB &= \begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}\begin{pmatrix}11&2\\4&5\\7&8\end{pmatrix} \\ &= \begin{pmatrix}40&36\\106&81\\172&126\end{pmatrix}\end{aligned}$$

It is often easier to remember this method by seeing the rows and columns highlighted, so let's look at another example. Imagine we want to find the following matrix product:

$$\begin{pmatrix}8&1&3&-2\\4&0&-6&1\end{pmatrix}\begin{pmatrix}3&1&2\\-1&2&3\\0&5&-4\\2&0&-2\end{pmatrix}$$

We begin by highlighting Row 1 of A with Column 1 of B, the result from finding the Scalar Product of these two highlighted vectors will go into Row 1, Column 1 of our answer:

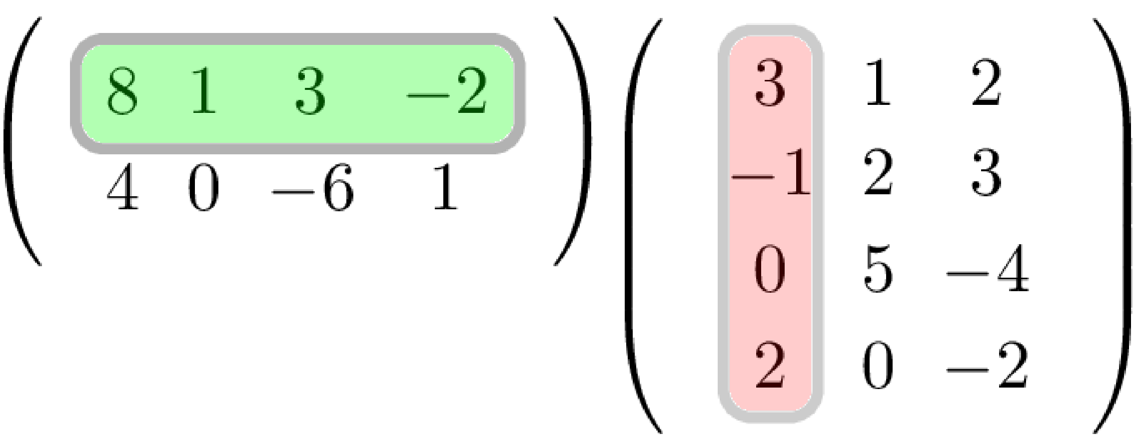

Figure 5.1: Highlighted rows and columns for matrix multiplication

This Scalar Product has value $8\times 3 + 1\times(-1) + 3\times 0 + (-2)\times 2 = 19$.

Next we highlight Row 1 of A with Column 2 of B, the result from finding the Scalar Product of these two highlighted vectors will go into Row 1, Column 2 of our answer:

$$\begin{pmatrix} 8 & 1 & 3 & -2 \\ 4 & 0 & -6 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 2 \\ -1 & 2 & 3 \\ 0 & 5 & -4 \\ 2 & 0 & -2 \end{pmatrix}$$

Figure 5.2: Highlighted rows and columns for matrix multiplication

This Scalar Product has value $8 \times 1 + 1 \times 2 + 3 \times 5 + (-2) \times 0 = 25$.

Finally we highlight Row 1 of A with Column 3 of B, the result from finding the Scalar Product of these two highlighted vectors will go into Row 1, Column 3 of our answer:

$$\begin{pmatrix} 8 & 1 & 3 & -2 \\ 4 & 0 & -6 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 2 \\ -1 & 2 & 3 \\ 0 & 5 & -4 \\ 2 & 0 & -2 \end{pmatrix}$$

Figure 5.3: Highlighted rows and columns for matrix multiplication

This Scalar Product has value $8 \times 2 + 1 \times 3 + 3 \times (-4) + (-2) \times (-2) = 11$.

These three calculations give (Row 1, Column 1), then (Row 1, Column 2) and (Row 1, Column 3) of the answer. We still need to repeat the above procedure but using Row 2 of A each time to find Row 2 values in our answer.

Practice
Calculate the three Scalar Products in the coloured images above, and then repeat the process but using Row 2 of A each time. Insert your six answers into their correct places to obtain your end result: the matrix product AB.
Verify that you get the following answer:

$$\begin{pmatrix} 8 & 1 & 3 & -2 \\ 4 & 0 & -6 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 2 \\ -1 & 2 & 3 \\ 0 & 5 & -4 \\ 2 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 19 & 25 & 11 \\ 14 & -26 & 30 \end{pmatrix}$$

If you get a different value in any element, go back and repeat the product. It is very common to make numerical mistakes when calculating matrix products.

When introducing matrix multiplication it was mentioned that you need each row of A to be of the same length as each column of B, this is actually not something you need to remember if you know the method! Notice that it would have been impossible to calculate the Scalar Products above unless every row of A matched each column of B for length!

A tip and a warning
If the two matrices match the correct shape requirement above then you can find the product using the method outlined.
If they don't match this shape requirement then **you cannot multiply** the matrices together.
Warning: The order of the matrix product matters! Which matrix was written on the left and which on the right will effect the answer. In fact it may not even be possible(!) to multiply the matrices, and even if you can you will normally get a totally different answer! So always make sure you know which matrix appears first in the product.

```
#> [Embedded question appears here in html version
#>  the hyperlink is provided below]
```

Link to Numbas on the web

In the examples provided above you were asked to perform matrix multiplications of matrices that were selected to have the correct shapes to make matrix multiplication possible. As a small exercise in checking what shape matrices can actually be multiplied together, here are a few examples to think about.

Here are three matrices, A, B and C:

$$A = \begin{pmatrix} 1 & 4 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 4 \\ 7 \\ 2 \end{pmatrix}, \quad C = \begin{pmatrix} 3 & 2 & -4 \\ 0 & 1 & 1 \end{pmatrix}$$

Question: Which of the six matrix products AB, AC, BC, BA, CA and CB are actually possible? If unsure, then refer back to the definition of matrix multiplication to think about what Scalar Products you need to calculate.
Hint: The length of each row in the left matrix, needs to match the length of each column in the right matrix.
Answers: [a]

[a] Permitted products are AC and CB

5.3 Special matrices, quantities and procedures

Matrices are a very rich area of maths, there are so many different matrices with so many uses that there are a few certain shapes and specific matrices which are given special names because they come up in numerous applications...

5.3.1 General pattern: Square matrices

A matrix is described as **square** if the number of rows matches the number of columns. The shape of the matrix grid will look like a square, it's as simple as that! Here are some examples of square matrices, of different sizes

$$\begin{pmatrix} 1 & 6 \\ -4 & -123 \end{pmatrix}, \quad \begin{pmatrix} 5 & 0 & 1 \\ 1 & 2 & 0 \\ -1 & 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 7 & 6 & -1 & 0 \\ 1 & -1 & 2 & 9 \\ 0 & 3 & -4 & 9 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

5.3.2 Special matrix: the zero matrix

There isn't just one *zero-matrix*, there is in fact one *zero-matrix* for every given possible matrix shape. Perhaps unsurprisingly the zero matrix of a particular shape is just a matrix where **every element is a zero**. e.g.

$$Z = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{ or } Z = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This matrix behaves a lot like the number 0 when used in matrix addition. Just like adding zero to a number makes no difference to a number, adding the zero matrix to a matrix makes no difference to the matrix.

5.3.3 Special matrix: the identity matrix – (see inverses later)

More useful than the *zero matrix* is the *identity matrix*, it is like the zero matrix above, but useful when doing matrix multiplication. The *identity matrix* can be described in three steps:

- it has to be a *square matrix* (see 5.3.1);
- all the elements along the diagonal from the top left to the bottom right are ones; and
- all the other elements are zeros.

So here are the *identity matrices* of shapes 2-by-2, 3-by-3 and 4-by-4 :

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.2}$$

The reason they are called *identity matrices* comes from the use of the word *identity* when describing the ordinary multiplication of a number by 1. The number 1 has the property that when you multiply a number by it then nothing changes, e.g. $13 \times 1 = 13$ and $1 \times x = x$ [5]. So while the zero matrix made no difference when adding (see 5.3.2), the idea with the identity matrix is the following:

[5] i.e. multiplying by 1 makes no difference

Multiplying a matrix by an identity matrix leaves it unchanged
Just like multiplying a number by 1 makes no difference to the number, multiplying a matrix by the identity matrix makes no difference to the matrix!
This is less easy to see, but **you should definitely try the following calculation for yourself**:

$$\begin{pmatrix} 3 & 1 & -1 \\ 8 & -2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 & -1 \\ 8 & -2 & 4 \end{pmatrix}$$

In algebra an identity matrix is represented by an I (a capital i), so the product above is written as $MI = M$. We say that for any matrix M, multiplying it by I (of the correct shape) gives you back M. If M is a square matrix, then you can easily check that also $IM = M$, so that you can put I on the left or the right of M and get the same answer. This property that $MI = IM$ is very special to an identity matrix I, because normally changing the order of the product of two matrices will give a totally different answer.

Note that we had to select an identity matrix of the correct size for the matrix multiplication to make sense. In this example we had to choose a matrix whose columns were length 3 to match with the length of each row of our starting matrix.

When working with identity matrices and algebra we normally use the letter I to describe the matrices, so the standard names of the matrices in (5.2) are actually

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

5.3.4 Special matrix: the transpose (and how to create it)

This one isn't quite a special matrix like the zero and identity matrices were, but it falls into the same category. Transposing is actually a procedure, something you can do *to* a matrix. To find the transpose of a matrix you just need to swap all the rows and columns.

The shape of the matrix will transform from an m-by-n matrix into an n-by-m matrix. e.g.

$$\begin{pmatrix} 1 & 8 & 0 \\ -1 & 3 & -5 \end{pmatrix} \text{ becomes } \begin{pmatrix} 1 & -1 \\ 8 & 3 \\ 0 & -5 \end{pmatrix},$$

i.e. Row 1 became Column 1, and Row 2 became Column 2. So our 2-by-3 matrix became a 3-by-2 matrix.

Interestingly the transpose of a square matrix will be another square matrix of the same shape.

Authors will use the letter T in a superscript as notation for the transpose of a matrix. So M becomes M^T, e.g.

$$\begin{pmatrix} 5 & 0 & 2 \\ 1 & -3 & 99 \\ 4 & 0 & 2 \end{pmatrix}^T = \begin{pmatrix} 5 & 1 & 4 \\ 0 & -3 & 0 \\ 2 & 99 & 2 \end{pmatrix}$$

An often useful fact about transposes is that if you transpose twice you will get back to where you started. It's true for any matrix M that

$$(M^T)^T = M.$$

Check it for yourself in the examples above.
You will also note that all the identity matrices (see Section 5.3.3) are unaffected by finding their transpose. e.g.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ i.e. } I_3^T = I_3$$

5.3.5 General pattern: Diagonal matrices

Like with identity matrices, we shall also use the adjective *diagonal* for square matrices. When describing identity matrices, every single element of the matrix was specified, either as a 1 (along the Northwest to Southeast diagonal), or a 0 elsewhere. We shall talk about that same diagonal from top left to bottom right when introducing *diagonal matrices*, but not every element will have a specified value.

A square matrix is called *diagonal* if *all the elements not on the diagonal from top left to bottom right are zero**. This definition may feel like a negative way of describing something, so an alternative more constructive description would be as follows:

In a *diagonal matrix* you can choose any values for the elements along the Northwest-to-Southeast diagonal, but all the other entries must be zero. Here are some examples,

$$\begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 7 \end{pmatrix}, \text{ and } \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}$$

You may notice that in second of these examples, one of the numbers on the diagonal was also zero, this is permitted. The definition merely demanded that the numbers not on the diagonal are all zero. In the third example, the letters a, b, c and d can take any values and the matrix will still be diagonal.

We sometimes refer to this diagonal from top left to bottom right as **the** *diagonal* of the matrix, even more technically as the *main diagonal*[6], i.e. the highlighted entries here:

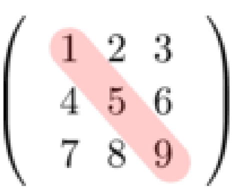

Figure 5.4: Illustration of the main diagonal of a matrix

Having named this as the *main diagonal of a matrix* then you can also define a *diagonal matrix* as one where the only place you might find non-zero elements is along the main diagonal.

It is clear that there are lots of possible diagonal matrices of a particular shape, because you can choose any numbers you wish to go along the main diagonal.

[6]some people call it also the *leading diagonal*

5.3.6 General pattern: Triangular matrices

These matrices are less important, and only used in more limited applications but we shall briefly mention them. There are actually two types of triangular matrices:

- A matrix is called *upper-triangular* if the **triangle of elements below the main diagonal** are all zero; and
- A matrix is called *lower-triangular* if the **triangle of elements above the main diagonal** are all zero.

This mismatch between “upper” and “below”, and between “lower” and “above” might sound like madness. However, it’s like our first definition of a diagonal matrix as one which is “zero in places not on the diagonal”. In fact there are two more natural and perfectly equivalent descriptions, namely

- A matrix is called *upper-triangular* if **only the triangle of elements including and above the main diagonal** are allowed to be non-zero;
- A matrix is called *lower-triangular* if **only the triangle of elements including and below the main diagonal** are allowed to be non-zero.

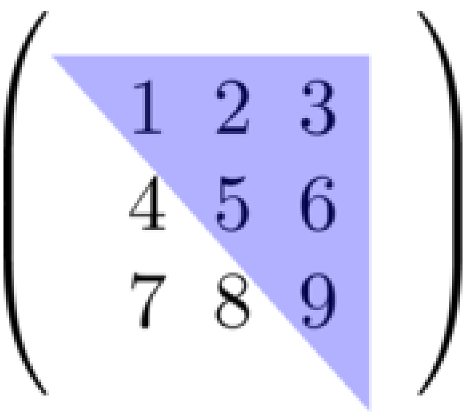

Figure 5.5: A picture of the upper-triangle highlighted, this matrix is not upper-triangular, because the entries below the triangle aren’t all zeros.

The single word *triangular* is used to describe either or both of these cases above. Just like you can talk of African elephants and Indian elephants as special examples of elephants.

We shan’t dwell on this adjective for now, but just show a few examples of triangular matrices. First a few upper-triangular ones,

$$\begin{pmatrix} 1 & 4 \\ 0 & -3 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 8 \\ 0 & 1 & -2 \\ 0 & 0 & 3 \end{pmatrix}$$

then a few lower-triangular ones,

$$\begin{pmatrix} 1 & 0 \\ 4 & -3 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & -1 & 2 \end{pmatrix}$$

finally a few that are **not** triangular at all,

$$\begin{pmatrix} 1 & 2 \\ 4 & -3 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 2 & -1 & 3 \\ 0 & 1 & 7 \\ 1 & 0 & 2 \end{pmatrix}$$

5.3.7 General pattern: Symmetric matrices

These types of matrices do arise a lot more in applications. They are also easy to identify just by looking at the entries of the matrix. The technical definition, which uses the word *transpose* from above, is to say that a matrix is *symmetric* if the transpose of the matrix is the same as the starting matrix.

The easier definition would be to say:

- the matrix has a square shape, and
- if you place a mirror down the main diagonal, then every number above the diagonal matches its image below the diagonal and vice-versa[7].

Here are some examples of symmetric matrices from which it will hopefully become obvious.

$$\begin{pmatrix} 5 & -1 & 2 \\ -1 & 3 & 4 \\ 2 & 4 & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & 0 & 8 & -4 \\ 0 & 3 & 7 & 11 \\ 8 & 7 & 2 & 9 \\ -4 & 11 & 9 & -2 \end{pmatrix} \tag{5.3}$$

Notice that along the main diagonal you can put whatever values you wish, but every other element has a mirror image element across the main diagonal.

In fact the matching mirror image location for the (i, j)-element is the one in position (j, i), e.g. Row 3 Column 1 must match Row 1 Column 3, and Row 2 Column 4 must match Row 4 Column 2.

5.3.8 Special quantity: The Determinant (important enough to have its own section below)

For every square matrix we can calculate a quantity known as the *determinant*. The process to calculate the determinant is easy for a 2-by-2 matrix but becomes progressively harder and messier for larger and larger matrices. We shall start with defining it for a 2-by-2 matrix.

For a 2-by-2 matrix which looks like this:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

its **determinant** is defined to be the value of

$$a \times d - b \times c$$

It's just the product of the main diagonal, minus the product of the other diagonal.
Notationally, the determinant of a matrix called M is normally written $\det(M)$.

Here is an image showing which pairs are multiplied, check the definition above to see which product is subtracted from the other product.

[7] the diagonal itself can take any values

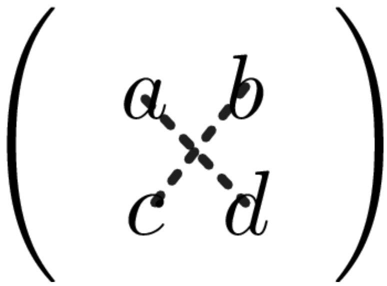

Figure 5.6: Illustration of products for finding a 2-by-2 deteminant.

Here are three example calculations, along with the det notation illustrated:

$$\det \begin{pmatrix} 4 & 7 \\ 1 & 2 \end{pmatrix} = 4 \times 2 - 7 \times 1 = 1$$

$$\det \begin{pmatrix} -1 & 2 \\ -3 & -4 \end{pmatrix} = (-1) \times (-4) - 2 \times (-3) = 4 + 6 = 10$$

$$\det \begin{pmatrix} 1 & 4 \\ -1 & -4 \end{pmatrix} = 1 \times (-4) - 4 \times (-1) = -4 + 4 = 0$$

Take extreme care when calculating determinants when negative numbers are around. You will frequently be subtracting negative numbers, and need to convert two negatives into a plus.

Now for the 3-by-3 determinant…

Calculating the determinant always involves using every element of the matrix, so when we get to 3-by-3 it's not the same pattern of diagonals. However, the formula for the 3-by-3 determinant can be described in terms of three 2-by-2 matrix determinants.[8]

Let's suppose we have a general 3-by-3 matrix that looks like this:

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}$$

We have named each element by its location to make what follows easier to understand.

Definition of the 3-by-3 matrix determinant
One way to calculate the 3-by-3 determinant is to evaluate:

$$a_{1,1} \times \det \begin{pmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{pmatrix} - a_{1,2} \times \det \begin{pmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{pmatrix} + a_{1,3} \times \det \begin{pmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{pmatrix}$$

[8] There are actually a few equivalent ways of describing the 3-by-3 determinant.

This calculation is most easily understood via three pictures, or via watching the subsequent video showing it in action. Geometrically, the 2-by-2 determinants required are the ones you are left with if you delete Row 1 and sequentially Columns 1, 2 and 3 from the matrix like this:

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}$$

Figure 5.7: The lower square is the first 2-by-2 determinant needed to find a 3-by-3 determinant

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}$$

Figure 5.8: The lower split square is the second 2-by-2 determinant needed to find a 3-by-3 determinant

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}$$

Figure 5.9: The lower square is the third 2-by-2 determinant needed to find a 3-by-3 determinant

Here is a video demonstration, which helps explain the procedure:

```
#> [Video appears here in html version
#>  the hyperlink is provided below]
```

Alternative direct video link: https://gcu.planetestream.com/View.aspx?id=5354~4r~SDdzPCeX

In a direct link to the Vectors section, we can now present a formula for the Vector Product of two vectors using this 3-by-3 determinant.

You may recall Equation (4.7) provided a general formula for the Vector Product, namely

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} bf - ce \\ cd - af \\ ae - bd \end{pmatrix} \tag{5.4}$$

Well, another way to write this is:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \det \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ a & b & c \\ d & e & f \end{pmatrix} \tag{5.5}$$

where we recall that $\underline{i}, \underline{j}, \underline{k}$ are just names for $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, respectively.

In words, we place our two given vectors into the second and third rows of a 3-by-3 matrix, and write the letters $\underline{i}, \underline{j}, \underline{k}$ on the top row. Then we can calculate the determinant, giving the answer in terms of $\underline{i}, \underline{j}$ and $\underline{k}$. The answer will be the Vector Product of the vectors visible in Row 2 and Row 3.

It is important to put the vectors into the matrix in the same order they appeared in the Vector Product, otherwise your answer will be incorrect[9]. The first vector in the Vector Product must go in Row 2, and the second vector in the Vector Product must go in Row 3.

Practice
Calculate the Vector Product of the two vectors from (4.8), namely

$$\underline{v} = \begin{pmatrix} 4 \\ 1 \\ -5 \end{pmatrix} \text{ and } \underline{w} = \begin{pmatrix} 2 \\ -9 \\ -3 \end{pmatrix}$$

by using the determinant formula above.
Check your answer with the answer calculated in Section 4.13 after Equation (4.8).
Warning: this can get numerically messy, and easy to make mistakes.

5.3.9 Revision exercises

This section contained a large number of new words, adjectives, nouns and terminology. So it will be useful to try a few exercises to revise this topic before moving on.

[9]you will get negative the correct answer if you put them the wrong way around

Examples to try
For each of the following matrices determine if it is an example of any of our special matrices, or fits any of the general patterns. As a reminder here are the key terms we are using: (we won't bother with triangular as it's not that useful)

(i) Square
(ii) Zero
(iii) Identity
(iv) Diagonal
(v) Symmetric

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 8 \end{pmatrix}$$

$$B = \begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix}$$

$$C = \begin{pmatrix} 2 & 3 & 7 \\ 0 & 4 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

We shall, in fact, go through these examples quickly to ensure some of the special cases are ironed out upon first study.

Answers, explained:
Square: A, B, D and E are all square matrices. Their number of rows matches their number of columns.
Zero: Only E is an example of a zero matrix. Remember, every element must be zero, not just some of them.
Identity: Only D is an example of an identity matrix, in this case it is the 3-by-3 identity, often called I_3.
Diagonal: B is a typical example of a diagonal matrix, all entries *not on the main diagonal* are zero. In fact D and E are technically also both diagonal – however observing they are an identity and zero matrix, respectively, provides even more information.
Symmetric: Here A is a typical example of a symmetric matrix. However, B, D and E are all technically examples too because they are each identical to their own transpose.

```
#> [Embedded question appears here in html version
#>  the hyperlink is provided below]
```

Link to Numbas on the web

5.4 The inverse of a matrix

When we work in standard algebra we often deal with equations that look like $8x = 3$, for which we divide both sides by 8 to reach $x = \frac{3}{8}$. This division by 8 is the equivalent to multiplying both sides by 8^{-1} because

$8 \times 8^{-1} = 1$. However, when we're working with matrices there is no easy way to define *division* by a matrix, but we can try. We shall only try and consider division/inversion by square matrices.

So if we have an equation like

$$AB = C$$

where A, B and C are matrices, and A is furthermore square-shaped, then there isn't a guaranteed way to divide both sides by A to get $B =$ something. The reason for this is that there isn't always a single matrix B that works in that equation[10].

Let's see an example. Suppose we have an equation like $AB = C$ of the form: (here B is a 2-by 2 matrix)

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} & B & \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 8 & 6 \end{pmatrix} \tag{5.6}$$

You would like to divide both sides by $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ to discover what B is. However, this isn't possible because there are infinitely many matrices B that work! For example, $B = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$ both work, and there are many more!

Sometimes it is possible, and there is a unique answer for B. This process of trying to divide by A in an equation like $AB = C$ to find B requires trying to find what is called the inverse of A.

The inverse of a (square) matrix
Firstly, we only try and do this for square matrices.
We use the notation A^{-1} for what we call the **inverse of matrix A**.
By direct comparison with 8^{-1} (the inverse of 8 being the number which multiplies 8 to give 1), a matrix Z is called the inverse of matrix A if

$$ZA = I$$

where I is an identity matrix. When such a matrix Z exists then we can write it as $Z = A^{-1}$, and we say that A is *invertible*[a].

[a] i.e. can be inverted

Warning: You cannot always find an inverse of a square matrix. Some square matrices don't have inverses, while for non-square matrices none of this even makes sense. Mathematically, however, the good news is that most square matrices can be inverted. Whether an inverse exists depends on the determinant being non-zero.

[10] if we could divide and get C/A then this is what B would be

Some examples of inverses:

If $A = \begin{pmatrix} 1 & 2 \\ 0 & 5 \end{pmatrix}$ then A does have an inverse and it is

$$A^{-1} = \begin{pmatrix} 1 & -\frac{2}{5} \\ 0 & \frac{1}{5} \end{pmatrix}.$$

If $A = \begin{pmatrix} 4 & 1 \\ 5 & 2 \end{pmatrix}$ then A does have an inverse and it is

$$A^{-1} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{5}{3} & \frac{4}{3} \end{pmatrix}.$$

You should check that in each of those examples, that

$$A^{-1}A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2.$$

Note: it is also always true that $AA^{-1} = I$ but we won't discuss this here.

You will note that in both of the examples above the inverse of A contained fractions with the same denominator, which actually turns out to be the determinant of A! This is because there is a general formula for the inverse of a 2-by-2 matrix and it's defined as follows:

Inverse of a 2-by-2 matrix formula
For any 2-by-2 matrix, A, if the determinant of A isn't zero (i.e. $\det(A) \neq 0$), then when

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

its inverse is

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \tag{5.7}$$

recalling that $\det(A) = ad - bc$.
Warning: This formula doesn't work if the determinant is zero.
In the case that $ad - bc = 0$ there is no inverse of A.

Now go back to Equation (5.6) and check that the A matrix is not invertible[11].

[11] because its determinant is zero

It is worth taking a moment to look at the formula for the inverse of a 2-by-2 matrix in Equation (5.7).
Notice the pattern:

- the main diagonal elements (the a and d) have been swapped;
- the off-diagonal elements (the b and c) have had their signs changed[a].

Then this result has been divided by the determinant.

[a] i.e. they've been multiplied by -1

Unfortunately this formula just needs to be memorized, and the general formula for the inverse of 3-by-3 matrices is much messier. The good news is that this quantity called the *determinant* can still be used as the deciding factor in all cases as to whether or not you can invert a matrix.

The determinant rule:

- You can always find the inverse of a matrix if its determinant is not zero.
- But if the determinant is zero, there is no inverse to find.

Why do we care about inverting a matrix?
Inverses will turn out to be really powerful. They can be used in a wide range of applications to solve equations involving matrices, because they allow us to perform simplifying algebra. Without them we are very limited in what we can do with algebra of matrices.

We shall consider the inverse of 3-by-3 and larger matrices in later sections (i.e. Sections 5.7 and 5.8).

First some examples to revise determinants, and inverses.

Practice
Below are four 2-by-2 matrices, in each case do the following:

- Calculate its determinant;
- Comment on whether the matrix will have an inverse;
- If an inverse exists, find it!
- After finding an inverse, multiply it by the original matrix to verify you get I_2.

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 9 \end{pmatrix}$$

$$B = \begin{pmatrix} -3 & -2 \\ 4 & 2 \end{pmatrix}$$

$$C = \begin{pmatrix} 2 & -8 \\ -1 & 4 \end{pmatrix}$$

$$D = \begin{pmatrix} 5 & 3 \\ 3 & 1 \end{pmatrix}$$

In more advanced courses on matrices you will see their applications. In the realm of 3D graphics and integration the determinant will turn out to have a geometric interpretation of *how much the world is scaled* when using a particular matrix. However we won't go into that topic here.

5.5 Properties of inverse matrices

Before moving on to some applications, we shall first see a few properties of inverses that can prove useful when performing algebra. These rules could be learned, or looked up when necessary. They don't need to be memorized, but you will learn them with practice. More important is to know there are some rules and when first working with matrices to go and check them.

Useful algebra rules for matrices

In all examples below A and B are general matrices, and k is a non-zero scalar (constant).

- The inverse of an inverse equals the original matrix, i.e.
$$(A^{-1})^{-1} = A.$$
- The inverse of the product of two matrices equals the product of their inverses **but in reverse order**! i.e.
$$(AB)^{-1} = B^{-1}A^{-1}.$$
- The inverse of a matrix multiplied by a scalar equals the inverse of the matrix multiplied by the inverse of the scalar, i.e.
$$(kA)^{-1} = \frac{1}{k}A^{-1}, \text{ note } k \neq 0.$$
- The inverse of the transpose of a matrix equals the transpose of the inverse of the matrix, i.e.
$$(A^T)^{-1} = (A^{-1})^T.$$

5.6 How to solve a 2-by-2 linear system using a matrix

Although we don't see highly practical real-world examples here, we can at least see a clever use of matrices to solve simultaneous equations. In the example presented below we limit it to just solving two simultaneous equations but, as we shall see later, the method can be applied to as many simultaneous equations as you wish.

Firstly, we want to see how matrices are related to simultaneous equations.

Consider the following two simultaneous equations:

$$\begin{aligned} x + 2y &= -1 \\ 4x - 3y &= 18 \end{aligned}$$

The clever use of matrices is to notice that there is a way to write this pair of equations as a single matrix equation.

How to re-write simultaneous equations in matrix format
Simultaneous equations can always be rewritten as a single matrix equation. For the example seen above it becomes,

$$\begin{pmatrix} 1 & 2 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 18 \end{pmatrix}$$

Expand out this single equation carefully and see it matches!

- Row 1 of the matrix multiplied by $\begin{pmatrix} x \\ y \end{pmatrix}$ equalling -1 yields the first simultaneous equation, and
- Row 2 of the matrix multiplied by $\begin{pmatrix} x \\ y \end{pmatrix}$ equalling 18 yields the second simultaneous equation.

To illustrate the general method, we can give a general formula.

General matrix method for solving simultaneous equations
Given a totally general pair of simultaneous equations:

$$\begin{aligned} ax + by &= g \\ cx + dy &= h \end{aligned}$$

where a, b, c, d, g and h are scalars; we wish to solve these equations to find the values of x and y which make both equations simultaneously true.
We can write the equations as a single matrix equation as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} g \\ h \end{pmatrix} \tag{5.8}$$

If we now let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then
if M^{-1} exists we can multiply (on the left) both sides of (5.8) by M^{-1} to obtain:

$$M^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = M^{-1} \begin{pmatrix} g \\ h \end{pmatrix} \tag{5.9}$$

however $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, so

$$M^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = M^{-1}M = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This means that Equation (5.9) turns into,

$$I_2 \begin{pmatrix} x \\ y \end{pmatrix} = M^{-1} \begin{pmatrix} g \\ h \end{pmatrix}. \tag{5.10}$$

We know that multiplying by I makes no difference, it's like multiplying a scalar by 1 (see Section

5.3.3), so this equation becomes

$$\begin{pmatrix} x \\ y \end{pmatrix} = M^{-1} \begin{pmatrix} g \\ h \end{pmatrix} \tag{5.11}$$

We now have an equation which tells us exactly what x and y are, which is our target!

Just before we see a worked example, a vital observation:

In all algebraic work, when solving equations with matrices, the key idea is almost always to **multiply both sides of the equation by the inverse of some matrix.**
This multiplication by an inverse will convert one of the products into an identity, and simplify the algebra. We saw this above, when trying to find a formula for $\begin{pmatrix} x \\ y \end{pmatrix}$ in (5.8), (5.9), (5.10) and (5.11).
Notice how the left of the equation changed between (5.8) and (5.11), that was all the result of multiplication by the correct inverse matrix.

Example
We shall solve our starting problem:

$$\begin{aligned} x + 2y &= -1 \\ 4x - 3y &= \ 18 \end{aligned}$$

We begin by writing it as:

$$\begin{pmatrix} 1 & 2 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 18 \end{pmatrix} \tag{5.12}$$

Next we let $M = \begin{pmatrix} 1 & 2 \\ 4 & -3 \end{pmatrix}$ and desire to find M^{-1}.
First we calculate $\det(M) = 1 \times (-3) - 2 \times 4 = -11$.
Second we use the general formula, from Equation (5.7)[a],

$$M^{-1} = -\frac{1}{11} \begin{pmatrix} -3 & -2 \\ -4 & 1 \end{pmatrix}$$

Third we perform **our key step**: we multiply (on the left) both sides by M^{-1} with the intention of cancelling out the existing M on the left.
So we multiply Equation (5.12) by M^{-1} to get

$$M^{-1} \begin{pmatrix} 1 & 2 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = M^{-1} \begin{pmatrix} -1 \\ 18 \end{pmatrix} \tag{5.13}$$

To see the algebra in gory detail we substitute our value of M^{-1}.
The *left side* looks like:

$$\text{Left} = -\frac{1}{11} \begin{pmatrix} -3 & -2 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

and the *right side* looks like:

$$\text{Right} = -\frac{1}{11}\begin{pmatrix}-3 & -2\\ -4 & 1\end{pmatrix}\begin{pmatrix}-1\\ 18\end{pmatrix}$$

and these two sides are equal.
However, $-\frac{1}{11}\begin{pmatrix}-3 & -2\\ -4 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\ 4 & -3\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, so Equation (5.13) becomes

$$I_2\begin{pmatrix}x\\ y\end{pmatrix} = -\frac{1}{11}\begin{pmatrix}-3 & -2\\ -4 & 1\end{pmatrix}\begin{pmatrix}-1\\ 18\end{pmatrix},$$

and $I_2\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}x\\ y\end{pmatrix}$, so we reach:

$$\begin{pmatrix}x\\ y\end{pmatrix} = -\frac{1}{11}\begin{pmatrix}-3 & -2\\ -4 & 1\end{pmatrix}\begin{pmatrix}-1\\ 18\end{pmatrix}.$$

These steps are the same every time we use this method[b]. The answer to our problem now results from performing the multiplication on the right.
You should check by hand that you agree with

$$-\frac{1}{11}\begin{pmatrix}-3 & -2\\ -4 & 1\end{pmatrix}\begin{pmatrix}-1\\ 18\end{pmatrix} = -\frac{1}{11}\begin{pmatrix}-33\\ 22\end{pmatrix}.$$

So,

$$\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}3\\ -2\end{pmatrix},$$

and so $x = 3$ and $y = -2$.

[a]Swap the main diagonal, change signs on the other diagonal
[b]though the numbers and matrices themselves obviously vary

This calculation might look like a lot of work, but it is actually the same fundamental algebra every time, except that the matrices M and M^{-1}, and the right-hand side will depend upon the problem being tackled. Here is the general shortcut answer...

Given simultaneous equations in the format:

$$M\underline{x} = \underline{b}$$

where
- M is a square matrix;
- $\underline{b}$ is a column vector containing the right-hand side values; and
- $\underline{x}$ is a column vector containing the unknown variables, e.g. (x, y), for which we wish to solve.

Then if M^{-1} exists[a] the answer is always

$$\underline{x} = M^{-1}\underline{b}.$$

[a]i.e. if M is invertible, i.e. it has a non-zero determinant

Practice
In each of the following problems convert the simultaneous equations into a matrix equation, then use the above "matrix inverse method" to solve to find x and y.
Note: It is easy to check your own answers, just substitute your solutions for x and y back into the original equations and confirm that they satisfy **both** equations.

- Problem 1: Solve
$$\begin{aligned} 4x + 2y &= 10 \\ 3x - 7y &= -1 \end{aligned}$$
- Problem 2: Solve
$$\begin{aligned} 3x - 5y &= 17 \\ -x - 6y &= 2 \end{aligned}$$

Warning:
In the case that the determinant of the 2-by-2 matrix is zero, then the matrix doesn't have an inverse. Therefore we do not have a matrix to multiply both sides of the equations by, as in Equation (5.9), to convert the left hand side into $I_2 \begin{pmatrix} x \\ y \end{pmatrix}$.
Typically there will be infinitely many possible (x, y) solutions to the equations. In very exceptional circumstances, however, there may be no solutions at all. We shall not go into these details here.

5.7 3-by-3 matrices

There is fundamentally nothing particularly different to working with 3-by-3 matrices than with 2-by-2 matrices. All the adjectives discussed in Section 5.3 were intentionally designed to work for matrices of all sizes[12].

Using 3-by-3 matrices allows you to model three-dimensional situations, so in one sense you may expect them to be more useful for real-world modelling. However, we very often find in Engineering applications that we are working on cross-sectional areas, or by symmetry we are able to neglect one dimension – so a great deal of applications you will see actually only require working in two-dimensions.

One key difference, however, is the application of matrices for solving simultaneous equations, where the dimensions of the square matrix correspond to the number of equations. Thus we need 3-by-3 matrices to be able to solve sets of 3 simultaneous equations, and indeed n-by-n matrices to solve n simultaneous equations (see Section 5.8 later).

The only real differences mathematically are that:
- the number of raw calculations required when multiplying 3-by-3 matrices is markedly increased;
- the formula for the determinant is more complicated; and
- there isn't a nice easy formula for the inverse of a 3-by-3 matrix, like there was for a 2-by-2 matrix in (5.7).

We actually already gave a formula, and procedure for the determinant of the 3-by-3 matrix in Section 5.3.8, where you will have seen it's considerably messier to calculate than finding a 2-by-2 determinant. However, whether a matrix has an inverse or not is still just a matter of whether its determinant is not zero or is zero.

[12]subject to some adjectives referring only to square matrices

There are a few different, equally good methods, for finding the inverse of a 3-by-3 matrix. In practice, the manual skill of finding such inverses is becoming less and less important in mathematics since software to find the inverse for you is readily available. As such these notes will **not** contain an explanation of the method. You are free to look up any good methods, these two are particularly popular and not too complicated:

- Co-factor and adjoint method for matrix inverses,
- Augmented matrix method for matrix inverses (also called the *Gauss-Jordan* method),

We shall discuss general computer methods briefly in the final Section 5.8.

For now, we shall just assume that we have a method for finding the inverse of a 3-by-3 matrix. The important fact you must know is that the inverse of a 3-by-3 matrix is a matrix which multiplied by the original matrix yields I_3, the 3-by-3 identity matrix, e.g.

$$\begin{pmatrix} 1 & 4 & 0 \\ 2 & -1 & 7 \\ 1 & 8 & 9 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 4 & 0 \\ 2 & -1 & 7 \\ 1 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Let us use a matrix inverse to solve a system of three simultaneous equations.

Solve these simultaneous equations using the inverse matrix method:

$$\begin{aligned} 3x + y + z &= 1 \\ x - 2y - z &= 0 \\ 8x - 2y + z &= -3 \end{aligned}$$

Assuming you already know that:

$$\begin{pmatrix} 3 & 1 & 1 \\ 1 & -2 & -1 \\ 8 & -2 & 1 \end{pmatrix}^{-1} = \frac{1}{7}\begin{pmatrix} 4 & 3 & -1 \\ 9 & 5 & -4 \\ -14 & -14 & 7 \end{pmatrix}.$$

The first step is to convert our simultaneous equations into a single matrix equation. The three equations will become a single 3-by-3 matrix multiplied by a 3-by-1 column matrix, equal to another 3-by-1 column matrix, like this:

$$\begin{pmatrix} 3 & 1 & 1 \\ 1 & -2 & -1 \\ 8 & -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}.$$

The next step is to find the inverse of our 3-by-3 matrix, which has already been done for us.
We then multiply both sides (on their lefts) by this inverse. So the left-hand side of the equation becomes:

$$\frac{1}{7}\begin{pmatrix} 4 & 3 & -1 \\ 9 & 5 & -4 \\ -14 & -14 & 7 \end{pmatrix}\begin{pmatrix} 3 & 1 & 1 \\ 1 & -2 & -1 \\ 8 & -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

and the right-hand side becomes

$$\frac{1}{7}\begin{pmatrix} 4 & 3 & -1 \\ 9 & 5 & -4 \\ -14 & -14 & 7 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix},$$

putting them together and simplifying we get…

$$I_3\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 4 & 3 & -1 \\ 9 & 5 & -4 \\ -14 & -14 & 7 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix},$$

and it just remains to perform the matrix multiplication on the right hand side…

$$\frac{1}{7}\begin{pmatrix} 4 & 3 & -1 \\ 9 & 5 & -4 \\ -14 & -14 & 7 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}.$$

So $x = 1$, $y = 3$ and $z = -5$.

Here is an example for you to try at home. Notice that you can just use the general result from Equation (5.6) once you have identified the necessary matrices in the question.

Practice
Solve these simultaneous equations using the inverse matrix method:

$$\begin{aligned} 4x + y + z &= 1 \\ x - 2y - z &= 0 \\ 8x - y + z &= -3 \end{aligned}$$

You should look online for a 3-by-3 matrix inverse calculator, to find the necessary inverse. As an exercise check that the supplied inverse is correct by multiplying the original matrix by its inverse to (hopefully) obtain the identity matrix I_3.

You can find many examples online if you wish to test your skills with this type of question, you will also find a couple of examples in the bonus exercises in Section 8.

5.8 Larger matrices

Larger matrices' main use is for representing large numbers of simultaneous equations in one single equation. To represent n simultaneous linear equations in the matrix format will involve designing an appropriate n-by-n matrix. Useful applications normally require calculating this matrix's inverse at some point.

Fortunately there are many efficient algorithms designed for computers to calculate inverses of large matrices. For humans it becomes extremely time consuming to calculate the inverse of matrices as they get larger, indeed humans will very rarely attempt to invert any matrix by hand which is larger than 3-by-3.

The only frequent exception to the previous claim about human calculation, is for certain special matrices whose inverses are actually easy to calculate. One interesting class of matrices like this are *diagonal matrices* (see Section 5.3.5). Recall that a diagonal matrix looks like this:

$$\begin{pmatrix} d_1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & d_2 & 0 & 0 & \ldots & 0 \\ 0 & 0 & d_3 & 0 & \ldots & 0 \\ 0 & 0 & 0 & d_4 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & d_n \end{pmatrix}$$

All entries not on the main diagonal are zeros, and the diagonal entries can take any values you like. Note: when working with large matrices we often use a sequence of three dots to say "continue the pattern".
A typical 4-by-4 example may look like this:

$$\begin{pmatrix} 7 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 10 \end{pmatrix}$$

Notice that finding the inverse is actually very easy. Just perform the following multiplication to see why,

$$\begin{pmatrix} \frac{1}{7} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & \frac{1}{10} \end{pmatrix} \begin{pmatrix} 7 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 10 \end{pmatrix},$$

the answer is just I_4, notice how all the zeros make the product quite easy to calculate.
So inverting an n-by-n *diagonal* matrix really is just as simple as going down the diagonal and *inverting* each element one at a time, i.e. replacing d with d^{-1} or $\frac{1}{d}$, however you prefer to write it. Just beware, this doesn't work if any elements on the diagonal are zeros!

As discussed earlier, the primary use of larger matrices is for solving systems of simultaneous equations, though they can be more complicated than standard linear equations. These methods can be used, for example, to solve systems of what are called differential equations (which you will study in later courses).

A general set of n simultaneous linear equations looks like this:

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + \ldots + a_{1,n}x_n &= b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 + \ldots + a_{2,n}x_n &= b_2 \\ a_{3,1}x_1 + a_{3,2}x_2 + a_{3,3}x_3 + \ldots + a_{3,n}x_n &= b_3 \\ \vdots \qquad \vdots \qquad \vdots \qquad\qquad \vdots \quad &= \vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + a_{n,3}x_3 + \ldots + a_{n,n}x_n &= b_n \end{aligned}$$

where all the a_{ij} and b_k are constants. And we had to name our variables x_1, x_2, x_3 etc... else we would have quickly run out of letters!
Such a system of equations can always be written in matrix format as:

$$A\underline{x} = \underline{b} \tag{5.14}$$

where

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{pmatrix}$$

and

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix}$$

The solution to Equation (5.14), assuming that $det(A) \neq 0$, so that A has an inverse[a], is always

$$\underline{x} = A^{-1}\underline{b}$$

The algebra to get from (5.14) to (5.8) has been seen twice before, first in Equations (5.9), (5.10), (5.11) for 2-by-2 matrices, and then again in (5.13) and (5.6) when working with 3-by-3 matrices. It always just involves multiplying on the left by the inverse of our square matrix, and simplifying.

[a]and so A^{-1} even exists!

Having glossed over the procedures for finding the inverse of a 3-by-3 matrix[13] we should conclude with some references to software you can use to find matrix inverses larger than 2-by-2 when needed.

Software for finding matrix inverses

- online you will find various websites called things like *matrix inverse calculator*;
- if using common spreadsheet software[a], there's a function called MINVERSE (matrix inverse);
- if using Wolfram Alpha you can type *inv*, *inverse* or *inverse of*;
- in Python one option is *numpy.linalg.inv()*;
- in R, you can use *inv* or *solve*;
- in Matlab you can use *inv*; and
- in other software, just search the web or the help files.

[a]Excel, OpenOffice or Sheets

Practice

Use one of the software options above to find the inverse of this 4-by-4 matrix:

$$\begin{pmatrix} 3 & 1 & 0 & 5 \\ 0 & 1 & -1 & 2 \\ 2 & 5 & -2 & 4 \\ 1 & 0 & -1 & 3 \end{pmatrix}$$

Verify your answer is the inverse by multiplying it by the matrix and verifying you get an identity matrix.

[13]though they were named

5.9 Summary of chapter

Having worked through this chapter and attempted the exercises you should now have developed understanding of the following topics:

- Know what a matrix is and describe its shape;
- Perform basic algebra including addition, subtraction and scalar multiplication;
- Determine if matrices are valid shapes to be multiplied together, and carry out the multiplication where appropriate;
- Calculate the determinant of 2-by-2 and 3-by-3 matrices;
- Calculate the inverse of 2-by-2 matrices, when it exists; and
- Use matrix methods to solve systems of linear equations.

Chapter 6

Algebra exercises

This section contains purely bonus exercises to accompany the Algebra chapter. Each chapter contained embedded examples and exercises, often with explanations.

These questions are provided with fewer explanations, generally just with the basic answers provided later. Of course, you can ask questions of the course lecturer about any problems here too.

6.1 Expansion of brackets and powers

1. Expand the following expressions involving products and powers
   i. $(3x^2)(5x^4)$
   ii. $(8x^3)(9x^5)$
   iii. $(4xy)(7x^2y^3)$
   iv. $(-x^3y^4)(6x^8y^9)$
   v. $(5x^2y^7)(-7x^3y^9)$
   vi. $(3ab)^2(-5ab^3)$
   vii. $(-5x^2y)(-8x^3y^2)$
   viii. $(-3a^b)(-4ab^3)$
   ix. $(4a)(ab)(-a^2b)$
   x. $(-3ab)(-4a^2b)(ab^2)^2$
2. Expand the following brackets and simplify
   i. $(x+2)(x+3)$
   ii. $(2x+5)(3x-7)$
   iii. $(2x-3)(2x+3)$
   iv. $(4y-8)(6y-13)$
   v. $(x+4)^2$
   vi. $(2x+3y)^2$
   vii. $(2x-3y)^2$
   viii. $(x+2)(x^2-4x+5)$
   ix. $(2x-1)(x^2-3x+7)$
   x. $(x+3)(x-4)(x+5)$
3. Expand the brackets $(a-b)(a+b)$ and then use this general formula to answer the following *without doing any new algebra*:
   i. Expand $(x+2)(x-2)$
   ii. Expand $(2x-3)(2x+3)$
   iii. Factorize x^2-16
   iv. Factorize $25x^2-64$

6.2 Simplifying fractions

1. Simplify the following fractions containing powers
   i.
   $$\frac{12a^2}{3a^2}$$
   ii.
   $$\frac{-32b^3}{8b^2}$$
   iii.
   $$\frac{42a^3b^4}{6a^2b}$$
   iv.
   $$\frac{72c^2d^3}{-9cd^2}$$
   v.
   $$\frac{4xy - 8xy^2}{2xy}$$
   vi.
   $$\frac{6\pi rh^2 + 18\pi r^2 h}{3\pi rh}$$

6.3 Exponentials and Logarithms

1. Use the log button your calculator (which is actually base 10, i.e. $\log_{10}$) and the $10^{\square}$ button to evaluate the following:

   i. $\log_{10}(1000)$
   ii. $\log_{10}(2.5)$
   iii. $10^{3.5}$
   iv. $\log_{10}(10^{3.5})$
   v. $\log_{10}(10^{-1.4})$
   vi. $10^{\log_{10}(6.1)}$
   vii. $10^{\log_{10}(0.5)}$

   Yes! The final four answers can be worked out by realizing that $10^{\square}$ and $\log_{10}(\square)$ are inverse operations. The first one you may realize the answer, if you notice that $1000 = 10^3$.

2. Use the ln (which is just $\log_e$) on your calculator, and its counterpart $e^{\square}$ to evaluate the following:

   i. $\ln(4.5)$
   ii. $\ln(2.75)$
   iii. $e^{0.6}$
   iv. $e^{-1.5}$
   v. $\ln(e^{0.6})$
   vi. $\ln(e^{-1.5})$
   vii. $e^{\ln(2.5)}$
   viii. $e^{\ln(0.75)}$

   Yes! The final four answers can be worked out by realizing that $e^{\square}$ and $\ln(\square)$ are inverse operations.

3. Use the three log laws to expand these logarithms into sums and differences of simpler logarithms.

   i. $\log_{10}\left(\frac{3x^2}{y}\right)$
   ii. $\ln\left(\frac{x^2y^2}{4}\right)$
   iii. $\log_{10}\left(\frac{100}{x+1}\right)$
   iv. $\ln\left(\frac{e^2}{2x+3}\right)$
   v. $\log_{10}\left(\sqrt{x^2+1}\right)$
   vi. $\ln\left(\sqrt{\frac{(x+1)^3(x-1)}{(x+2)}}\right)$

4. Combine these sums and differences of simple logs into one single logarithm:

   i. $3\log_{10}(x) + 2\log_{10}(y) - 4\log_{10}(z)$
   ii. $2\log_{10}(x+y) - \frac{1}{2}\log_{10}(z)$
   iii. $3\ln(x) + \frac{1}{3}\ln(y)$
   iv. $4\ln(2x+y) - 2\ln(z)$

6.4 Re-arranging formula to make a variable the subject

1. For each of the following formulae, change the subject of the equation to the quantity indicated in the bracket on the right:
   i. $v = u + at, \qquad (t)$
   ii. $s = \frac{1}{2}at^2, \qquad (t)$
   iii. $s = ut + \frac{1}{2}at^2, \qquad (u)$
   iv. $s = ut + \frac{1}{2}at^2, \qquad (a)$
   v. $v = \sqrt{u^2 + 2as}, \qquad (s)$
   vi. $v = \sqrt{u^2 + 2as}, \qquad (u)$
   vii. $y = a + bx^3, \qquad (x)$
   viii. $i = 5e^t, \qquad (t)$
   ix. $i = 8e^{-2t}, \qquad (t)$
   x. $y = 10^{2x+1}, \qquad (x)$
   xi. $y = 10^{3x-2}, \qquad (x)$
   xii. $y = c_0 + a_0e^{-kt}, \qquad (t)$
2. For each of the following, change the subject of the equation to the quantity indicated in the bracket on the right:
   i.
   $$y = \frac{2-x}{3+x}, \qquad (x)$$
   ii.
   $$y = \frac{4+2x}{5-x}, \qquad (x)$$
   iii.
   $$z = \frac{xy}{x+y}, \qquad (y)$$
   iv.
   $$C = \frac{C_1C_2}{C_1 - C_2}, \qquad (C_2)$$

6.5 Factorizing quadratics

1. Factorize each of the following quadratic expressions:

i. $x^2 + 3x + 2$
ii. $x^2 + 5x + 4$
iii. $x^2 + 4x + 4$
iv. $x^2 + x - 2$
v. $x^2 - x - 2$
vi. $x^2 + 5x - 6$
vii. $x^2 - 5x - 6$
viii. $x^2 + x - 6$
ix. $x^2 - x - 6$
x. $2x^2 + 11x + 12$

6.6 Solving equations – variety of learned methods

1. Solve the following equations: where answers are decimals, give accurate to five decimal places

i. $4x + 5 = 8$
ii. $10x - 8 = -12$
iii. $6x + 3 = 2x - 5$
iv. $3x - 9 = 5x + 2$
v. $5 + 3x^2 = 32$
vi. $5x^3 + 320 = 0$
vii. $e^x = 0.75$
viii. $e^{4x} = 0.2$
ix. $5e^{-x} - 8 = 7$
x. $4 + 12e^{-3x} = 13$
xi. $\ln(3x) = -2$
xii. $5\ln(2x) + 3 = 0$
xiii. $5 - 3\ln(4x) = -10$
xiv. $\ln(6x + 4) = 0.25$
xv. $10^x = 2.5$
xvi. $10^{3x-2} = 1.75$
xvii. $\log_{10}(4x) = 0.32$
xviii. $\log_{10}(5x + 4) = 0.65$
xix. $2^{4x-1} = 5$
xx. $3^{2x+4} = 6$

2. Solve the following quadratic equations by factorization, then repeat using the quadratic formula and check your answers match.

i. $x^2 + 2x - 15 = 0$
ii. $x^2 - 9x + 20 = 0$
iii. $x^2 + 10x + 25 = 0$
iv. $2x^2 - 7x - 4 = 0$

3. Complete the square on each of these quadratic equations. Then try solving them using the quadratic formula, what happens?

i. $x^2 + 2x + 2 = 0$
ii. $x^2 + 4x - 9 = 0$
iii. $x^2 + 4x + 9 = 0$
iv. $2x^2 - 3x + 4 = 0$

Now look back at your completed squared versions, can you see why they couldn't be solved?

4. Solve the following pairs of simultaneous equations:

i.

$$\begin{aligned} 4x + 3y &= 2 \\ 2x - y &= 16 \end{aligned}$$

ii.

$$\begin{aligned} 5x + 2y &= 1 \\ 4x + 3y &= -2 \end{aligned}$$

iii.

$$\begin{aligned} 6x - 2y &= -20 \\ 4x + 5y &= -7 \end{aligned}$$

iv.

$$\begin{aligned} 8x - 5y &= 24.5 \\ 2x - 3y &= 10.5 \end{aligned}$$

Chapter 7

Vectors exercises

This section contains purely bonus exercises to accompany the Vectors chapter. Each chapter contained embedded examples and exercises, often with explanations.

These questions are provided with fewer explanations, generally just with the basic answers provided later. Of course, you can ask questions of the course lecturer about any problems here too.

7.1 Vector sketching

1. Sketch the following vectors (given in polar format) starting from the specified points:
   i. $(2, 30°)$ starting from $(-3, 2)$.
   ii. $(5, 30°)$ starting from $(1, 4)$.
   iii. $(0.25, 200°)$ starting from $(5, 4)$.
   iv. $(6, -30°)$ starting from $(-2, 2)$.
2. Sketch the following vectors (given in the rectangular format) starting from the specified points:
   i. $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ starting from $(2, 1)$.
   ii. $\begin{pmatrix} -4 \\ 2 \end{pmatrix}$ starting from $(2, -4)$.
   iii. $\begin{pmatrix} 6.5 \\ 3.5 \end{pmatrix}$ starting from $(-4, -4)$.
   iv. $\begin{pmatrix} -2.5 \\ -4.5 \end{pmatrix}$ starting from $(-6, 2)$.

7.2 Vector format conversion

1. Convert the following vectors to rectangular (Cartesian) format without using a calculator (Hint: the angles are nice):
   i. $(4, 0°)$
   ii. $(8, 90°)$
   iii. $(6, 180°)$
   iv. $(5, 270°)$
   v. $(3, -90°)$
   vi. $(10, , -180°)$
2. Convert the following vectors into rectangular (Cartesian) format with the aid of the standard equations (4.1), and then your calculator.
   i. $(2, 30°)$

ii. $(3, 80^\circ)$
iii. $(1, 120^\circ)$
iv. $(5, 315^\circ)$
v. $(4, 200^\circ)$
vi. $(2, -75^\circ)$

3. Convert the following vector to Polar format, without the use of a calculator (Hint: the angles will be nice)
   i. $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$
   ii. $\begin{pmatrix} 0 \\ 10 \end{pmatrix}$
   iii. $\begin{pmatrix} -8 \\ 0 \end{pmatrix}$
   iv. $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$
4. Convert the following vectors from rectangular format to Polar format,
   i. $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$
   ii. $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$
   iii. $\begin{pmatrix} -4 \\ -3 \end{pmatrix}$
   iv. $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$
   v. $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$
   vi. $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

7.3 Vector addition, subtraction and multiplication

1. Given the vectors $\underline{v}_1 = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$ and $\underline{v}_3 = \begin{pmatrix} -5 \\ -1 \end{pmatrix}$, determine the following vectors. In cases (i)-(vi) you may like to sketch the answers graphically too.
   i. $\underline{v}_1 + \underline{v}_2$
   ii. $\underline{v}_1 + \underline{v}_2 + \underline{v}_3$
   iii. $\underline{v}_1 + (-\underline{v}_2)$
   iv. $\underline{v}_1 - \underline{v}_3$
   v. $2\underline{v}_1$
   vi. $3\underline{v}_2$
   vii. $2\underline{v}_1 + 3\underline{v}_2$
   viii. $4\underline{v}_1 - 2\underline{v}_2 + 5\underline{v}_3$

7.4 Scalar Products and Relative Positions

1. For each of the following pairs of vectors, determine their lengths $|\underline{v}_1|$, $|\underline{v}_1|$, their Scalar Product $(\underline{v}_1 \cdot \underline{v}_2)$, and the angle between them, θ.
   i. $\underline{v}_1 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$
   ii. $\underline{v}_1 = \begin{pmatrix} -4 \\ 6 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} 5 \\ -8 \end{pmatrix}$
   iii. $\underline{v}_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
   iv. $\underline{v}_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
2. A triangle has vertices $A = (1, 1)$, $B = (4, 2)$ and $C = (3, 4)$.
   i. Sketch this triangle on the standard xy-axes (including the origin).
   ii. Determine the relative position vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.
   iii. **By calculation (using a Scalar Product)**, determine the angle of the triangle at the vertex A.
   iv. If you have access to a protractor, try measuring the angle in your diagram to see if it agrees.
3. A triangle has vertices $A = (-2, 1)$, $B = (3, 1)$ and $C = (1, 5)$.
   i. Draw this triangle accurately (including the origin).
   ii. **By calculation (using a Scalar Product)**, determine the angle of the triangle at the vertex B.
   iii. If you have access to a protractor, try measuring the angle in your diagram to see if it agrees.

7.5 Scalar Product further applications

1. For each of these pairs of vectors, determine whether the pair are orthogonal to each other (i.e. at right-angles):
   i. $\underline{v}_1 = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$
   ii. $\underline{v}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$
   iii. $\underline{v}_1 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} -5 \\ 1 \end{pmatrix}$
   iv. $\underline{v}_1 = \begin{pmatrix} 15 \\ 3 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} -2 \\ 10 \end{pmatrix}$
   v. $\underline{v}_1 = \begin{pmatrix} 10 \\ 3 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} -10 \\ 33 \end{pmatrix}$
2. For each of the following forces $\underline{F}$ find the Scalar projection of $\underline{F}$ in the direction of the given vector $\underline{d}$:
   i. $\underline{F} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$, $\underline{d} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$
   ii. $\underline{F} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$, $\underline{d} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
   iii. $\underline{F} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$, $\underline{d} = \begin{pmatrix} 0.6 \\ 0.8 \end{pmatrix}$

iv. $\underline{F} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\underline{d} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ (Note, you will need to scale $\underline{d}$ to length 1 first)

3. For each of the following forces $\underline{F}$ find the work done in moving an object from location P to location Q:
   i. $\underline{F} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$, $P = (1,3), Q = (2,5)$.
   ii. $\underline{F} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$, $P = (1,3), Q = (-2,-5)$.
   iii. $\underline{F} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $P = (-6,-4), Q = (0,0)$.

7.6 3-dimensional vectors (mixture of topics)

1. Given the three 3-dimensional vectors

$$\underline{v}_1 = \begin{pmatrix} 4 \\ -5 \\ 1 \end{pmatrix}, \underline{v}_2 = \begin{pmatrix} 5 \\ 6 \\ -3 \end{pmatrix}, \underline{v}_3 = \begin{pmatrix} 6 \\ 2 \\ -4 \end{pmatrix}$$

   determine the following:
   i. $\underline{v}_1 + \underline{v}_2 + \underline{v}_3$
   ii. $4\underline{v}_1 + 3\underline{v}_2 - 2\underline{v}_3$
   iii. $\underline{v}_1 + (-\underline{v}_3)$
   iv. $-2\underline{v}_1 + \underline{v}_2 + 5\underline{v}_3$
   v. $\underline{v}_1 \cdot \underline{v}_2$
   vi. $\underline{v}_2 \cdot \underline{v}_3$
   vii. $\underline{v}_1 \times \underline{v}_2$
   viii. $\underline{v}_2 \cdot \underline{v}_3$
   ix. $\underline{v}_3 \cdot \underline{v}_2$
2. Using the same vectors from the previous question, find the following: (you may use previous answers, to speed up calculations)
   i. The angle between $\underline{v}_1$ and $\underline{v}_2$.
   ii. The angle between $\underline{v}_1$ and $\underline{v}_3$.
   iii. The component/projection of $\underline{v}_1$ in the direction of $\underline{v}_2$. (Careful here to use a length 1 vector)
   iv. The component/projection of $\underline{v}_3$ in the direction of $\underline{v}_1$. (Careful here to use a length 1 vector)
3. A triangle in 3D has vertices at $A = (1,4,3)$, $B = (4,2,0)$ and $C = (5,4,6)$.
   i. Determine the relative position vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.
   ii. Determine the angle of the triangle at vertex A.
4. A triangle in 3D has vertices at $A = (-2,3,-5)$, $B = (2,0,4)$ and $C = (1,5,1)$
   i. Determine the angle of the triangle at vertex C.

7.7 3D vector applications

1. A force $\underline{F} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$ acts on a particle which moves from point $P = (1,4,-1)$ to point $Q = (-2,3,1)$.
   i. Determine the displacement vector $\underline{d}$ of the particle.
   ii. Determine the *work done* by the force.

2. A force $\underline{F}$ acts through a point P. Given that $P = (2, 3, -5)$, $Q = (1, 2, -3)$ and $\underline{F} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$,

   i. Calculate the moment of this force about the point Q. You may want to consult Section 4.14.2.

Chapter 8

Matrices exercises

This section contains purely bonus exercises to accompany the Matrices chapter. Each chapter contained embedded examples and exercises, often with explanations.

These questions are provided with fewer explanations, generally just with the basic answers provided later. Of course, you can ask questions of the course lecturer about any problems here too.

8.1 Matrix algebra

1. Consider the following matrices:

$$A = \begin{pmatrix} 1 & 0 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & -1 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$$

$$D = \begin{pmatrix} -3 & 1 \\ 6 & 5 \end{pmatrix}, \quad E = \begin{pmatrix} -2 & 6 & 3 \\ 1 & 0 & -1 \\ 5 & 8 & 4 \end{pmatrix}, \quad F = \begin{pmatrix} 9 & 0 & 7 \\ 2 & -2 & 0 \\ 1 & 6 & 5 \end{pmatrix}$$

$$G = \begin{pmatrix} 2 & 6 \\ 1 & 3 \\ -1 & 5 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & -6 \\ -1 & 0 \\ 3 & 8 \end{pmatrix}$$

$$J = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}, \quad K = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

   i. State the shape of each matrix.
   ii. Determine the following algebraic combinations (not all may be possible!)
      a. $A + B$
      b. $C - D$
      c. $E + F$
      d. $E - F$
      e. $2G + 3H$
      f. $3C - D$
      g. GC
      h. EJ
      i. JE
      j. CD

k. DC
l. EF
m. HE
n. AK
o. KA

2. Simplify the following linear combinations:
   i. $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} -1 & -2 \\ 2 & 1 \end{pmatrix}$
   ii. $\begin{pmatrix} 1 & 3 \\ \frac{1}{2} & -2 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} & 0 \\ \frac{3}{2} & 1 \end{pmatrix}$
   iii. $3\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$
   iv. $2\begin{pmatrix} 1 & 4 \\ -1 & 6 \end{pmatrix} + 3\begin{pmatrix} 0 & -2 \\ 4 & 1 \end{pmatrix}$
3. Calculate the following matrix products:
   i. $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -1 & -2 \\ 2 & 1 \end{pmatrix}$
   ii. $\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -3 \\ 1 & 2 & -2 \end{pmatrix}$
   iii. $\begin{pmatrix} 1 & \frac{1}{2} & 3 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ -1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$
   iv. $\begin{pmatrix} 3 & -2 & 5 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ 7 \end{pmatrix}$

8.2 Matrix properties

1. Calculate the following products:
   i. $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\left[\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -1 & -2 \\ 2 & 1 \end{pmatrix}\right]$
   ii. $\left[\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -1 & -2 \\ 2 & 1 \end{pmatrix}\right]\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$
   iii. Look at the two calculations you have just performed, are the answers the same? What does this tell you about the order or matrix multiplication?
2. Answer the following two questions about squaring matrices:
   i. Which of the following matrices can be squared? $A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ and $M = \begin{pmatrix} 1 & 0 & -3 \\ 1 & 2 & -2 \end{pmatrix}$
   ii. In general, considering all possible matrices, which matrices can be squared?
3. For the matrix $A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$,
   i. Calculate A^T and $(A^T)^T$.
   ii. How does $(A^T)^T$ compare to A?

4. Let $A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$ and $B = \begin{pmatrix} u & v & w \\ x & y & z \end{pmatrix}$.
   i. Evaluate $A^T + B^T$ and $(A + B)^T$.
   ii. Comment on your answer.

8.3 Matrix determinants and inverses

1. Calculate the determinants of each of the following matrices:
   i. $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
   ii. $\begin{pmatrix} 3 & -2 \\ 4 & 5 \end{pmatrix}$
   iii. $\begin{pmatrix} 6 & -3 \\ -4 & 2 \end{pmatrix}$
   iv. $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$
   v. $\begin{pmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 2 & 3 & 2 \end{pmatrix}$
   vi. $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 4 & 4 \end{pmatrix}$
2. For each matrix in the previous question, determine its inverse or explain why you know it doesn't have an inverse. (You may use a computer for 3-by-3 matrix inverses)
3. For the matrix $A = \begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix}$, use the standard formula to determine its inverse (written A^{-1}) and calculate the two products AA^{-1} and $A^{-1}A$. Comment on your results.
4. For the matrix $A = \begin{pmatrix} 4 & 1 & 3 \\ 2 & 1 & 2 \end{pmatrix}$ calculate the product AA^T. Can you always multiply a matrix by its own transpose?
5. Let $A = \begin{pmatrix} 2 & 5 \\ 1 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & -5 \\ 0 & 3 \end{pmatrix}$.
   i. First calculate A^{-1} and B^{-1}
   ii. Next evaluate, by hand, $AB, (AB)^{-1}$ and $B^{-1}A^{-1}$
   iii. Comment on whether you were expecting these final two evaluations to be equal
6. Consider the following matrix $A = \begin{pmatrix} 1 & 2 \\ k & 3 \end{pmatrix}$ where k is a constant.
   i. Determine which value(s) of k allow A to be invertible.

ii. Calculate the inverse of A (note your answer will contain k)

7. Let $A = \begin{pmatrix} 1 & -2 & 1 \\ -3 & 2 & -1 \\ 2 & -1 & 0 \end{pmatrix}$.

   i. Calculate the determinant of A.
   ii. Find the inverse of A (using your computer).
   iii. By hand, calculate the matrix products AA^{-1} and $A^{-1}A$.
   iv. Did your results agree with what you were expecting?

8. Given that $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix}$, find the inverse matrix D^{-1} without using a computer.

9. (Hardest) Let $A = \begin{pmatrix} 4 & 0 & 9 \\ 0 & 5+k & -3 \\ 0 & 2 & k \end{pmatrix}$ be a matrix, where k is a constant.

   i. Calculate the determinant of A (your answer will contain k).
   ii. Use your determinant formula to determine all values of k when the matrix is invertible.

8.4 Solving simultaneous of equations

1. Consider the following system of simultaneous equations:

$$\begin{aligned} 2x - 5y &= 2 \\ 3x - 7y &= 1 \end{aligned}$$

   i. Express these simultaneous equations in matrix form, i.e. as $A\underline{x} = \underline{b}$, where A is a square matrix, and both $\underline{b}$ and $\underline{x}$ are single column matrices.
   ii. Determine the inverse, A^{-1}, of your matrix A.
   iii. Use A^{-1} to find the solution to the simultaneous equations.

2. A system of equations is given by

$$\begin{aligned} 2x - y + 3z &= 13 \\ x - 2y - 3z &= -4 \\ 4x - 2y - 3z &= 8 \end{aligned}$$

   i. Express these simultaneous equations in the matrix form

$$A\underline{x} = \underline{b}.$$

   ii. Determine the matrix A^{-1} (use a computer, it will exist in this case).
   iii. Use your result in the previous part to solve this system of equations for x, y, z.

Chapter 9

Algebra exercises – Solutions

This section contains solutions to the bonus exercises to accompany the Algebra chapter. Each chapter contained embedded examples and exercises, often with explanations.

These solutions are provided with fewer explanations, generally just with the basic answers provided later. Of course, you can ask questions of the course lecturer about any problems here too.

9.1 Expansion of brackets and powers

1. Expand the following expressions involving products and powers
   i. $(3x^2)(5x^4) = 15x^6$
   ii. $(8x^3)(9x^5) = 72x^8$
   iii. $(4xy)(7x^2y^3) = 28x^3y^4$
   iv. $(-x^3y^4)(6x^8y^9) = -18x^{11}y^{13}$
   v. $(5x^2y^7)(-7x^3y^9) = -35x^5y^{16}$
   vi. $(3ab)^2(-5ab^3) = -45a^3b^3$
   vii. $(-5x^2y)(-8x^3y^2) = 40x^5y^3$
   viii. $(-3a^b)(-4ab^3) = 12a^3b^4$
   ix. $(4a)(ab)(-a^2b) = -4a^4b^2$
   x. $(-3ab)(-4a^2b)(ab^2)^2 = 12a^5b^6$
2. Expand the following brackets and simplify
   i. $(x+2)(x+3) = x^2+5x+6$
   ii. $(2x+5)(3x-7) = 6x^2+x-35$
   iii. $(2x-3)(2x+3) = 4x^2-9$
   iv. $(4y-8)(6y-13) = 24y^2-100y+104$
   v. $(x+4)^2 = x^2+8x+16$
   vi. $(2x+3y)^2 = 4x^2+12xy+9y^2$
   vii. $(2x-3y)^2 = 4x^2-12xy+9y^2$
   viii. $(x+2)(x^2-4x+5) = x^3-2x^2-3x+10$
   ix. $(2x-1)(x^2-3x+7) = 2x^3-7x^2+17x-7$
   x. $(x+3)(x-4)(x+5) = x^3+4x^2-17x-60$
3. Expand the brackets $(a-b)(a+b)$ and then use this general formula to answer the following *without doing any new algebra* $(a-b)(a+b) = a^2-b^2$ so…
   i. $(x+2)(x-2) = x^2-4$
   ii. $(2x-3)(2x+3) = 4x^2-9$
   iii. $x^2-16 = (x-4)(x+4)$
   iv. $25x^2-64 = (5x-8)(5x+8)$

9.2 Simplifying fractions

1. Simplify the following fractions containing powers

   i.
   $$\frac{12a^2}{3a^2} = 4$$

   ii.
   $$\frac{-32b^3}{8b^2} = -4b$$

   iii.
   $$\frac{42a^3b^4}{6a^2b} = 7ab^3$$

   iv.
   $$\frac{72c^2d^3}{-9cd^2} = -8cd$$

   v.
   $$\frac{4xy - 8xy^2}{2xy} = 2 - 4y$$

   vi.
   $$\frac{6\pi rh^2 + 18\pi r^2 h}{3\pi rh} = 2h + 6r$$

9.3 Exponentials and Logarithms

1. Use the log button your calculator (which is actually base 10, i.e. $\log_{10}$) and the $10^{\square}$ button to evaluate the following:

   i. $\log_{10}(1000) = 3$
   ii. $\log_{10}(2.5) = 0.397940009$
   iii. $10^{3.5} = 3162.27766$
   iv. $\log_{10}(10^{3.5}) = 3.5$
   v. $\log_{10}(10^{-1.4}) = -1.4$
   vi. $10^{\log_{10}(6.1)} = 6.1$
   vii. $10^{\log_{10}(0.5)} = 0.5$

   Could you have known the answer to any of these without using your calculator?

2. Use the ln (which is just $\log_e$) on your calculator, and its counterpart $e^{\square}$ to evaluate the following:

   i. $\ln(4.5) = 1.504077397$
   ii. $\ln(2.75) = 1.011600912$
   iii. $e^{0.6} = 1.822118800$
   iv. $e^{-1.5} = 0.223130160$
   v. $\ln(e^{0.6}) = 0.6$
   vi. $\ln(e^{-1.5}) = -1.5$
   vii. $e^{\ln(2.5)} = 2.5$
   viii. $e^{\ln(0.75)} = 0.75$

   Could you have known the answer to any of these without using your calculator?

3. Use the three log laws to expand these logarithms into sums and differences of simpler logarithms.

   i. $\log_{10}\left(\frac{3x^2}{y}\right) = \log_{10}(3) + 2\log_{10}(x) - \log_{10}(y)$
   ii. $\ln\left(\frac{x^2y^2}{4}\right) = 2\ln(x) + 2\ln(y) - \ln(4)$
   iii. $\log_{10}\left(\frac{100}{x+1}\right) = 2 - \log_{10}(x+1)$
   iv. $\ln\left(\frac{e^2}{2x+3}\right) = 2 - \ln(2x+3)$
   v. $\log_{10}\left(\sqrt{x^2+1}\right) = \frac{1}{2}\log_{10}(x^2+1)$
   vi. $\ln\left(\sqrt{\frac{(x+1)^3(x-1)}{(x+2)}}\right) = \frac{1}{2}\left[3\ln(x+1) + \ln(x-1) - \ln(x+2)\right]$

4. Combine these sums and differences of simple logs into one single logarithm:

   i. $3\log_{10}(x) + 2\log_{10}(y) - 4\log_{10}(z) = \log_{10}\left(\frac{x^3y^2}{z^4}\right)$
   ii. $2\log_{10}(x+y) - \frac{1}{2}\log_{10}(z) = \log_{10}\left(\frac{(x+y)^2}{\sqrt{z}}\right)$
   iii. $3\ln(x) + \frac{1}{3}\ln(y) = \ln\left(x^3y^{\frac{1}{3}}\right)$
   iv. $4\ln(2x+y) - 2\ln(z) = \ln\left(\frac{(2x+y)^4}{z^2}\right)$

9.4 Re-arranging formula to make a variable the subject

1. For each of the following formulae, change the subject of the equation to the quantity indicated in the bracket on the right:
   i. $t = \frac{v-u}{a}$
   ii. $t = \sqrt{\frac{2s}{a}}$
   iii. $u = \frac{s-\frac{1}{2}at^2}{t}$ or $s = \frac{s}{t} - \frac{1}{2}at$
   iv. $a = \frac{2(s-ut)}{t^2}$
   v. $s = \frac{v^2-u^2}{2a}$
   vi. $u = \sqrt{v^2 - 2as}$
   vii. $x = \left(\frac{y-a}{b}\right)^{\frac{1}{3}}$
   viii. $\ln(\frac{i}{5})$ or $\ln(i) - \ln(5)$
   ix. $t = -\frac{1}{2}\ln(i/8)$ or $-\frac{1}{2}\ln(i) + \frac{1}{2}\ln(8)$
   x. $x = \frac{1}{2}\left(\log_{10} - 1\right)$
   xi. $x = \frac{1}{3}\left(\log_{10}(y) + 2\right)$
   xii. $t = -\frac{1}{k}\ln\left(\frac{y-c_0}{a_0}\right)$
2. For each of the following, change the subject of the equation to the quantity indicated in the bracket on the right:
   i. $y = \frac{2-x}{3+x}, \quad \Longrightarrow x = \frac{2-3y}{y+1}$
   ii. $y = \frac{4+2x}{5-x}, \quad \Longrightarrow x = \frac{5y-4}{y+2}$
   iii. $z = \frac{xy}{x+y}, \quad \Longrightarrow y = \frac{xz}{x-z}$
   iv. $C = \frac{C_1C_2}{C_1-C_2}, \quad \Longrightarrow C_2 = \frac{CC_1}{C+C_1}$

9.5 Factorizing quadratics

1. Factorize each of the following quadratic expressions:

i. $x^2 + 3x + 2 = (x+1)(x+2)$
ii. $x^2 + 5x + 4 = (x+1)(x+4)$
iii. $x^2 + 4x + 4 = (x+2)^2$
iv. $x^2 + x - 2 = (x+2)(x-1)$
v. $x^2 - x - 2 = (x-2)(x+1)$
vi. $x^2 + 5x - 6 = (x+6)(x-1)$
vii. $x^2 - 5x - 6 = (x-6)(x+1)$
viii. $x^2 + x - 6 = (x+3)(x-2)$
ix. $x^2 - x - 6 = (x-3)(x+2)$
x. $2x^2 + 11x + 12 = (2x+3)(x+4)$

9.6 Solving equations – variety of learned methods

1. Solve the following equations: where answers are decimals, give accurate to five decimal places

   i. $4x + 5 = 8 \Longrightarrow x = \frac{3}{4}$
   ii. $10x - 8 = -12 \Longrightarrow x = -\frac{2}{5}$
   iii. $6x + 3 = 2x - 5 \Longrightarrow x = -2$

iv. $3x - 9 = 5x + 2 \implies x = -\frac{11}{2}$
v. $5 + 3x^2 = 32 \implies x = 3$ and $x = -3$
vi. $5x^3 + 320 = 0 \implies x = 4$
vii. $e^x = 0.75 \implies x = \ln(0.75) = -0.28768$
viii. $e^{4x} = 0.2 \implies x = \frac{1}{4}\ln(0.2) = -0.40236$
ix. $5e^{-x} - 8 = 7 \implies -\ln(3) = -1.09861$
x. $4 + 12e^{-3x} = 13 \implies x = -\frac{1}{3}\ln(0.75) = 0.09589$
xi. $\ln(3x) = -2 \implies x = \frac{1}{3}e^{-2} = 0.04511$
xii. $5\ln(2x) + 3 = 0 \implies x = \frac{1}{2}e^{-0.6} = 0.27441$
xiii. $5 - 3\ln(4x) = -10 \implies x = \frac{1}{4}e^{5} = 37.10329$
xiv. $\ln(6x + 4) = 0.25 \implies x = \frac{1}{6}\left(e^{0.25} - 4\right) = -0.45266$
xv. $10^x = 2.5 \implies x = \log_{10}(2.5) = 0.39794$
xvi. $10^{3x-2} = 1.75 \implies x = \frac{1}{3}\left(\log_{10}(1.75) + 2\right) = 0.74768$
xvii. $\log_{10}(4x) = 0.32 \implies x = \frac{1}{4}10^{0.32} = 0.52232$
xviii. $\log_{10}(5x + 4) = 0.65 \implies x = \frac{1}{5}\left(10^{0.65} - 4\right) = 0.09337$
xix. $2^{4x-1} = 5 \implies x = \frac{1}{4}\left(\frac{\log(5)}{\log(2)} + 1\right) = 0.83048$
xx. $3^{2x+4} = 6 \implies x = \frac{1}{2}\left(\frac{\log(6)}{\log(3)} - 4\right) = -1.18454$ any choice of log base is fine.

2. Solve the following quadratic equations by factorization, then repeat using the quadratic formula and check your answers match.

i. $x^2 + 2x - 15 = (x + 5)(x - 3) \implies x = -5$ and $x = 3$
ii. $x^2 - 9x + 20 = (x - 5)(x - 4) \implies x = 4$ and $x = 5$
iii. $x^2 + 10x + 25 = (x + 5)^2 \implies x = -5$ (repeated)
iv. $2x^2 - 7x - 4 = (2x + 1)(x - 4) \implies x = -\frac{1}{2}$ and $x = 4$

3. Complete the square on each of these quadratic equations. Then try solving them using the quadratic formula, what happens?

i. $x^2 + 2x + 2 = (x + 1)^2 + 1$. No solution possible.
ii. $x^2 + 4x - 9 = (x + 2)^2 - 13$. Solutions are $x = -2 + \sqrt{13}$ and $x = -2 - \sqrt{13}$.
iii. $x^2 + 4x + 9 = (x + 2)^2 + 5$. No solution possible.
iv. $2x^2 - 3x + 4 = 2((x - \frac{3}{4})^2 + \frac{1}{4}) = 2(x - \frac{3}{4})^2 + \frac{1}{4}$. No solution possible.

In all cases where there was no (real) solution, the number outside the completed square bracket was positive. This means that if we tried to re-arrange to make x the subject we would need to square root a negative number, which we cannot do. This is because when you square a number the answer is never negative.

4. Solve the following pairs of simultaneous equations:

i.

$$\begin{aligned} 4x + 3y &= 2 \\ 2x - y &= 16 \end{aligned}$$

$\implies x = 5, y = -6$

ii.

$$\begin{aligned} 5x + 2y &= 1 \\ 4x + 3y &= -2 \end{aligned}$$

$\implies x = 1, y = -2$

iii.

$$\begin{aligned} 6x - 2y &= -20 \\ 4x + 5y &= -7 \end{aligned}$$

$\implies x = -3, y = 1$

iv.

$$8x - 5y = 24.5$$
$$2x - 3y = 10.5$$

$\implies x = 1.5, y = -2.5$

Chapter 10

Vectors exercises - Solutions

This section contains solutions to the bonus exercises to accompany the Vectors chapter. Each chapter contained embedded examples and exercises, often with explanations.

These solutions are provided with fewer explanations, generally just with the basic answers provided later. Of course, you can ask questions of the course lecturer about any problems here too.

10.1 Vector sketching

1. Sketch the following vectors (given in polar format) starting from the specified points:
   i. $(2, 30^\circ)$ starting from $(-3, 2)$.
   ii. $(5, 30^\circ)$ starting from $(1, 4)$.
   iii. $(0.25, 200^\circ)$ starting from $(5, 4)$.
   iv. $(6, -30^\circ)$ starting from $(-2, 2)$.

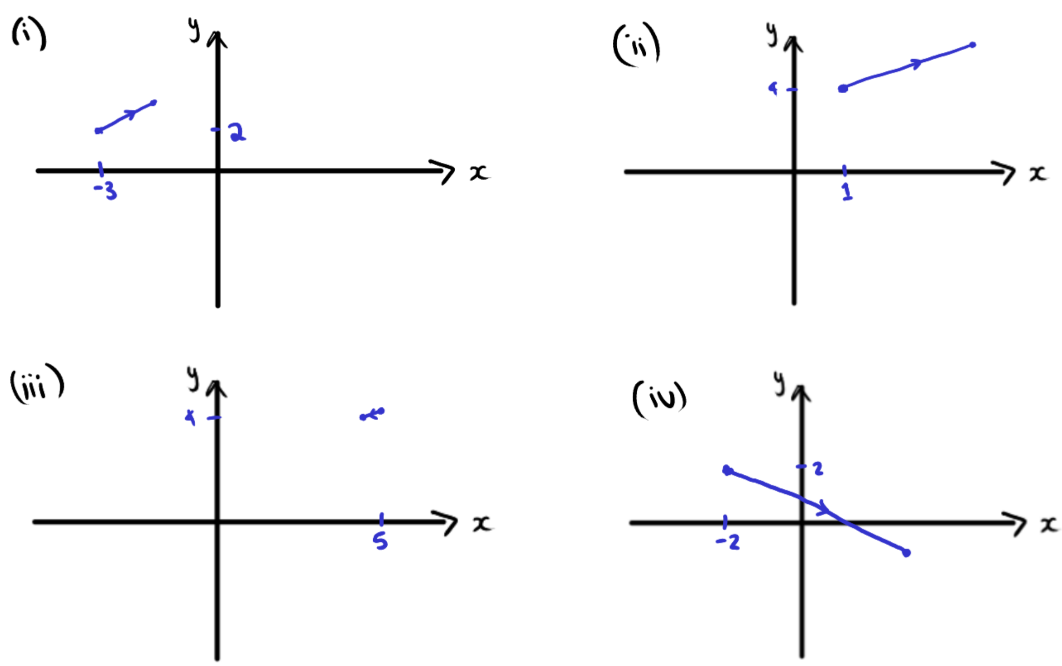

Figure 10.1: Sketches for question 1

2. Sketch the following vectors (given in the rectangular format) starting from the specified points:

i. $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ starting from $(2, 1)$.

ii. $\begin{pmatrix} -4 \\ 2 \end{pmatrix}$ starting from $(2, -4)$.

iii. $\begin{pmatrix} 6.5 \\ 3.5 \end{pmatrix}$ starting from $(-4, -4)$.

iv. $\begin{pmatrix} -2.5 \\ -4.5 \end{pmatrix}$ starting from $(-6, 2)$.

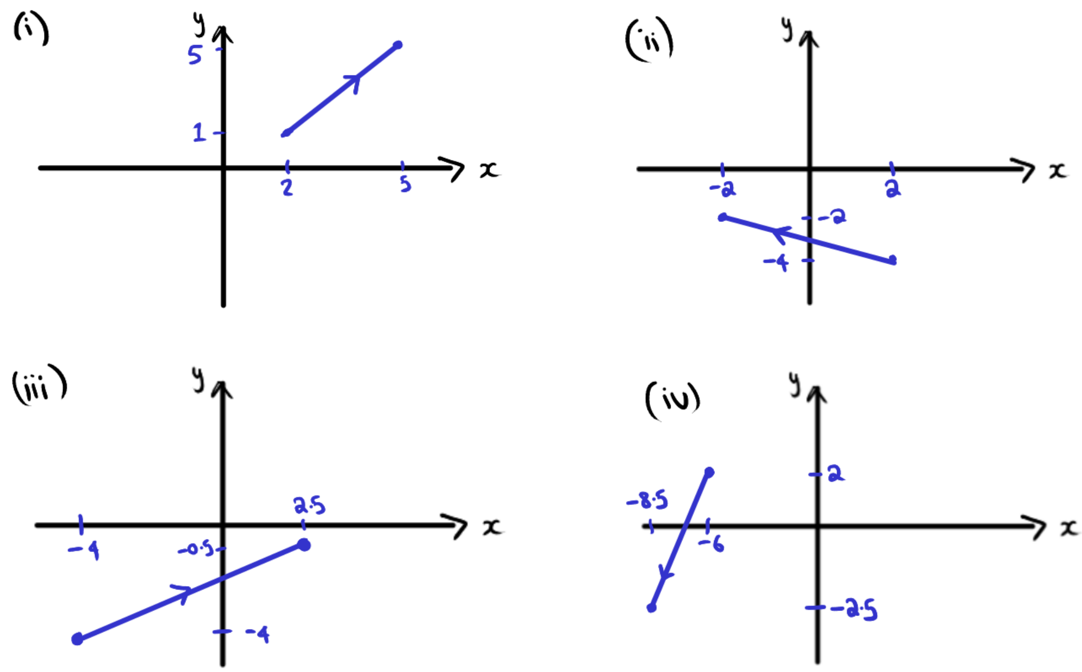

Figure 10.2: Sketches for question 2

10.2 Vector format conversion

1. Convert the following vectors to rectangular (Cartesian) format without using a calculator (Hint: the angles are nice):

   i. $(4, 0°) \longrightarrow \begin{pmatrix} 4 \\ 0 \end{pmatrix}$

   ii. $(8, 90°) \longrightarrow \begin{pmatrix} 0 \\ 8 \end{pmatrix}$

   iii. $(6, 180°) \longrightarrow \begin{pmatrix} -6 \\ 0 \end{pmatrix}$

   iv. $(5, 270°) \longrightarrow \begin{pmatrix} 0 \\ -5 \end{pmatrix}$

v. $(3, -90^\circ) \longrightarrow \begin{pmatrix} 0 \\ -3 \end{pmatrix}$

vi. $(10, , -180^\circ) \longrightarrow \begin{pmatrix} -10 \\ 0 \end{pmatrix}$

2. Convert the following vectors into rectangular (Cartesian) format with the aid of the standard equations (4.1), and then your calculator.

i. $(2, 30^\circ) \longrightarrow \begin{pmatrix} 1.732 \\ 1 \end{pmatrix}$

ii. $(3, 80^\circ) \longrightarrow \begin{pmatrix} 0.521 \\ 2.954 \end{pmatrix}$

iii. $(1, 120^\circ) \longrightarrow \begin{pmatrix} -0.5 \\ 0.866 \end{pmatrix}$

iv. $(5, 315^\circ) \longrightarrow \begin{pmatrix} 3.536 \\ -3.536 \end{pmatrix}$

v. $(4, 200^\circ) \longrightarrow \begin{pmatrix} -3.759 \\ -1.368 \end{pmatrix}$

vi. $(2, -75^\circ) \longrightarrow \begin{pmatrix} 0.518 \\ -1.932 \end{pmatrix}$

3. Convert the following vector to Polar format, without the use of a calculator (Hint: the angles will be nice)

i. $\begin{pmatrix} 5 \\ 0 \end{pmatrix} \longrightarrow (5, 0^\circ)$

ii. $\begin{pmatrix} 0 \\ 10 \end{pmatrix} \longrightarrow (10, 90^\circ)$

iii. $\begin{pmatrix} -8 \\ 0 \end{pmatrix} \longrightarrow (8, 180^\circ)$

iv. $\begin{pmatrix} 0 \\ -2 \end{pmatrix} \longrightarrow (2, -90^\circ)$ or $(2, 270^\circ)$

4. Convert the following vectors from rectangular format to Polar format,

i. $\begin{pmatrix} -3 \\ 2 \end{pmatrix} \longrightarrow (3.606, 146.310^\circ)$

ii. $\begin{pmatrix} 3 \\ -2 \end{pmatrix} \longrightarrow (3.606, -33.690^\circ)$

iii. $\begin{pmatrix} -4 \\ -3 \end{pmatrix} \longrightarrow (5, -143.130^\circ)$

iv. $\begin{pmatrix} 4 \\ 3 \end{pmatrix} \longrightarrow (5, 36.870^\circ)$

v. $\begin{pmatrix} -2 \\ 1 \end{pmatrix} \longrightarrow (2.236, 153.435^\circ)$

vi. $\begin{pmatrix} 2 \\ -1 \end{pmatrix} \longrightarrow (2.236, -26.565°)$

10.3 Vector addition, subtraction and multiplication

1. Given the vectors $\underline{v}_1 = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$ and $\underline{v}_3 = \begin{pmatrix} -5 \\ -1 \end{pmatrix}$, determine the following vectors. In cases (i)-(vi) you may like to sketch the answers graphically too.

   i. $\underline{v}_1 + \underline{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

   ii. $\underline{v}_1 + \underline{v}_2 + \underline{v}_3 = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$

   iii. $\underline{v}_1 + (-\underline{v}_2) = \begin{pmatrix} 6 \\ -7 \end{pmatrix}$

   iv. $\underline{v}_1 - \underline{v}_3 = \begin{pmatrix} 9 \\ -2 \end{pmatrix}$

   v. $2\underline{v}_1 = \begin{pmatrix} 8 \\ -6 \end{pmatrix}$

   vi. $3\underline{v}_2 = \begin{pmatrix} -6 \\ 12 \end{pmatrix}$

   vii. $2\underline{v}_1 + 3\underline{v}_2 = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$

   viii. $4\underline{v}_1 - 2\underline{v}_2 + 5\underline{v}_3 = \begin{pmatrix} -5 \\ -25 \end{pmatrix}$

10.4 Scalar Products and Relative Positions

1. For each of the following pairs of vectors, determine their lengths $|\underline{v}_1|$, $|\underline{v}_1|$, their Scalar Product $(\underline{v}_1 \cdot \underline{v}_2)$, and the angle between them, θ.

   i. $\underline{v}_1 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

$$|\underline{v_1}| = \sqrt{8}, |\underline{v_2}| = \sqrt{10}, \underline{v_1} \cdot \underline{v_2} = -4, \theta = 116.57°$$

   ii. $\underline{v}_1 = \begin{pmatrix} -4 \\ 6 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} 5 \\ -8 \end{pmatrix}$

$$|\underline{v_1}| = \sqrt{52}, |\underline{v_2}| = \sqrt{89}, \underline{v_1} \cdot \underline{v_2} = -68, \theta = 178.32°$$

   iii. $\underline{v}_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$|\underline{v_1}| = 5, |\underline{v_2}| = 1, \underline{v_1} \cdot \underline{v_2} = 3, \theta = 53.13°$$

iv. $\underline{v}_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$|\underline{v_1}| = 5, |\underline{v_2}| = 1, \underline{v_1} \cdot \underline{v_2} = 4, \theta = 36.87^\circ$$

2. A triangle has vertices $A = (1, 1)$, $B = (4, 2)$ and $C = (3, 4)$.
   i. Sketch this triangle on the standard xy-axes (including the origin).
   ii. Determine the relative position vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$\overrightarrow{AB} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

   iii. **By calculation (using a Scalar Product)**, determine the angle of the triangle at the vertex A.

$$\theta = 37.9^\circ$$

   iv. If you have access to a protractor, try measuring the angle in your diagram to see if it agrees.
3. A triangle has vertices $A = (-2, 1)$, $B = (3, 1)$ and $C = (1, 5)$.
   i. Draw this triangle accurately (including the origin).
   ii. **By calculation (using a Scalar Product)**, determine the angle of the triangle at the vertex B.

$$\theta = 63.4^\circ$$

   iii. If you have access to a protractor, try measuring the angle in your diagram to see if it agrees.

10.5 Scalar Product further applications

1. For each of these pairs of vectors, determine whether the pair are orthogonal to each other (i.e. at right-angles): **Vectors are orthogonal if and only if their Scalar Product is zero**:
   i. $\underline{v}_1 = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \Longrightarrow \underline{v_1} \cdot \underline{v_2} = 0 \Rightarrow$ Orthogonal
   ii. $\underline{v}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \Longrightarrow \underline{v_1} \cdot \underline{v_2} = -5 \Rightarrow$ Not orthogonal
   iii. $\underline{v}_1 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} -5 \\ 1 \end{pmatrix} \Longrightarrow \underline{v_1} \cdot \underline{v_2} = 0 \Rightarrow$ Orthogonal
   iv. $\underline{v}_1 = \begin{pmatrix} 15 \\ 3 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} -2 \\ 10 \end{pmatrix} \Longrightarrow \underline{v_1} \cdot \underline{v_2} = 0 \Rightarrow$ Orthogonal
   v. $\underline{v}_1 = \begin{pmatrix} 10 \\ 3 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} -10 \\ 33 \end{pmatrix} \Longrightarrow \underline{v_1} \cdot \underline{v_2} = -1 \Rightarrow$ Not orthogonal
2. For each of the following forces $\underline{F}$ find the Scalar projection of $\underline{F}$ in the direction of the given vector $\underline{d}$: **In each case we need to calculate $\underline{F} \cdot \hat{\underline{d}}$ where $\hat{\underline{d}}$ is a vector of length 1 in the direction of d. In the first three examples $\hat{\underline{d}} = \underline{d}$ is already of length 1. In the final case we need to divide by its length (which is 5)**
   i. $\underline{F} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$, $\underline{d} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \Longrightarrow \underline{F} \cdot \hat{\underline{d}} = 5/\sqrt{2} = 3.536$
   ii. $\underline{F} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$, $\underline{d} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Longrightarrow \underline{F} \cdot \hat{\underline{d}} = -3$
   iii. $\underline{F} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$, $\underline{d} = \begin{pmatrix} 0.6 \\ 0.8 \end{pmatrix} \Longrightarrow \underline{F} \cdot \hat{\underline{d}} = 0.2$

iv. $\underline{F} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\underline{d} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \Longrightarrow \underline{F} \cdot \hat{\underline{d}} = \frac{1}{5}10 = 2$

3. For each of the following forces $\underline{F}$ find the work done in moving an object from location P to location Q: **In each case we need to calculate $\underline{F} \cdot \overrightarrow{PQ}$

i. $\underline{F} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$, $P = (1, 3), Q = (2, 5) \Rightarrow \underline{F} \cdot \overrightarrow{PQ} = 9.$

ii. $\underline{F} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$, $P = (1, 3), Q = (-2, -5) \Rightarrow \underline{F} \cdot \overrightarrow{PQ} = 11.$

iii. $\underline{F} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $P = (-6, -4), Q = (0, 0) \Rightarrow \underline{F} \cdot \overrightarrow{PQ} = 24.$

10.6 3-dimensional vectors (mixture of topics)

1. Given the three 3-dimensional vectors

$$\underline{v}_1 = \begin{pmatrix} 4 \\ -5 \\ 1 \end{pmatrix}, \ \underline{v}_2 = \begin{pmatrix} 5 \\ 6 \\ -3 \end{pmatrix}, \ \underline{v}_3 = \begin{pmatrix} 6 \\ 2 \\ -4 \end{pmatrix}$$

determine the following:

i. $\underline{v}_1 + \underline{v}_2 + \underline{v}_3 = \begin{pmatrix} 15 \\ 3 \\ -6 \end{pmatrix}$

ii. $4\underline{v}_1 + 3\underline{v}_2 - 2\underline{v}_3 = \begin{pmatrix} 19 \\ -6 \\ 3 \end{pmatrix}$

iii. $\underline{v}_1 + (-\underline{v}_3) = \begin{pmatrix} -2 \\ -7 \\ -5 \end{pmatrix}$

iv. $-2\underline{v}_1 + \underline{v}_2 + 5\underline{v}_3 = \begin{pmatrix} 27 \\ 26 \\ -25 \end{pmatrix}$

v. $\underline{v}_1 \cdot \underline{v}_2 = -13$

vi. $\underline{v}_2 \cdot \underline{v}_3 = 54$

vii. $\underline{v}_1 \times \underline{v}_2 = \begin{pmatrix} 9 \\ 17 \\ 49 \end{pmatrix}$

viii. $\underline{v}_2 \times \underline{v}_3 = \begin{pmatrix} -18 \\ 2 \\ -26 \end{pmatrix}$

ix. $\underline{v}_3 \times \underline{v}_2 = \begin{pmatrix} 18 \\ -2 \\ 26 \end{pmatrix}$

2. Using the same vectors from the previous question, find the following: (you may use previous answers, to speed up calculations)
   i. The angle between $\underline{v}_1$ and $\underline{v}_2$ is $\theta = 103.87°$.
   ii. The angle between $\underline{v}_1$ and $\underline{v}_3$ is $\theta = 78.1°$.
   iii. The component/projection of $\underline{v}_1$ in the direction of $\underline{v}_2$ is -3.153.
   iv. The component/projection of $\underline{v}_3$ in the direction of $\underline{v}_1$ is 1.543.
3. A triangle in 3D has vertices at $A = (1, 4, 3)$, $B = (4, 2, 0)$ and $C = (5, 4, 6)$.
   i. Determine the relative position vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$\overrightarrow{AB} = \begin{pmatrix} 3 \\ -2 \\ -3 \end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$$

   ii. Determine the angle of the triangle at vertex A.

$$\theta = 82.65°$$

4. A triangle in 3D has vertices at $A = (-2, 3, -5)$, $B = (2, 0, 4)$ and $C = (1, 5, 1)$
   i. Determine the angle of the triangle at vertex C.

$$\theta = 105.40°$$

10.7 3D vector applications

1. A force $\underline{F} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$ acts on a particle which moves from point $P = (1, 4, -1)$ to point $Q = (-2, 3, 1)$.
   i. Determine the displacement vector $\underline{d}$ of the particle.

$$\underline{d} = \overrightarrow{PQ} = \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix}$$

   ii. Determine the *work done* by the force.

$$\text{Work done} = \underline{F} \cdot \underline{d} = 3.$$

2. A force $\underline{F}$ acts through a point P. Given that $P = (2, 3, -5)$, $Q = (1, 2, -3)$ and $\underline{F} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$,
   i. Calculate the moment of this force about the point Q. You may want to consult Section 4.14.2.

$$\text{Moment of force} = \underline{M} = \underline{r} \times \underline{F} = \begin{pmatrix} -8 \\ 6 \\ -1 \end{pmatrix}$$

   A moment/torque produces a rotation. The direction of this vector gives the direction of the axis of that rotation. The length is the strength of the force.

Chapter 11

Matrices exercises - Solutions

This section contains solutions to the bonus exercises to accompany the Introduction to Matrices chapter. Each chapter contained embedded examples and exercises, often with explanations.

These solutions are provided with fewer explanations, generally just with the basic answers provided later. Of course, you can ask questions of the course lecturer about any problems here too.

11.1 Matrix algebra

1. Consider the following matrices:

$$A = \begin{pmatrix} 1 & 0 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & -1 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$$

$$D = \begin{pmatrix} -3 & 1 \\ 6 & 5 \end{pmatrix}, \quad E = \begin{pmatrix} -2 & 6 & 3 \\ 1 & 0 & -1 \\ 5 & 8 & 4 \end{pmatrix}, \quad F = \begin{pmatrix} 9 & 0 & 7 \\ 2 & -2 & 0 \\ 1 & 6 & 5 \end{pmatrix}$$

$$G = \begin{pmatrix} 2 & 6 \\ 1 & 3 \\ -1 & 5 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & -6 \\ -1 & 0 \\ 3 & 8 \end{pmatrix}$$

$$J = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}, \quad K = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

   i. State the shape of each matrix.
      (A) 1-by-3
      (B) 1-by-3
      (C) 2-by-2
      (D) 2-by-2
      (E) 3-by-3
      (F) 3-by-3
      (G) 3-by-2
      (H) 3-by-2
      (I) 3-by-1
      (J) 3-by-1
   ii. Determine the following algebraic combinations (not all may be possible!)

a. $A+B=\begin{pmatrix}5 & -1 & -1\end{pmatrix}$

b. $C-D=\begin{pmatrix}4 & 1\\ -2 & -2\end{pmatrix}$

c. $E+F=\begin{pmatrix}7 & 6 & 10\\ 3 & -2 & -1\\ 6 & 14 & 9\end{pmatrix}$

d. $E-F=\begin{pmatrix}-11 & 6 & -4\\ -1 & 2 & -1\\ 4 & 2 & -1\end{pmatrix}$

e. $2G+3H=\begin{pmatrix}4 & -6\\ -1 & 6\\ 7 & 34\end{pmatrix}$

f. $3C-D=\begin{pmatrix}6 & 5\\ 6 & 4\end{pmatrix}$

g. $GC=\begin{pmatrix}26 & 22\\ 13 & 11\\ 19 & 13\end{pmatrix}$

h. $EJ=\begin{pmatrix}2\\ -2\\ 18\end{pmatrix}$

i. $JE=$ Not possible, not correct shapes.

j. $CD=\begin{pmatrix}9 & 11\\ 6 & 19\end{pmatrix}$

k. $DC=\begin{pmatrix}1 & -3\\ 26 & 27\end{pmatrix}$

l. $EF=\begin{pmatrix}-3 & 6 & 1\\ 8 & -6 & 2\\ 65 & 8 & 55\end{pmatrix}$

m. HE Not possible, not correct shapes.

n. $AK=\begin{pmatrix}-1\end{pmatrix}$. This one is weird, but it's a 1-by-1 matrix!

o. $KA=\begin{pmatrix}-1 & 0 & 3\\ 2 & 0 & -6\\ 0 & 0 & 0\end{pmatrix}$. This is extremely weird, every Scalar Product was a product of just two numbers. You rarely do this kind of product.

2. Simplify the following linear combinations:

i. $\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}+\begin{pmatrix}-1 & -2\\ 2 & 1\end{pmatrix}=\begin{pmatrix}0 & -2\\ 4 & 2\end{pmatrix}$

ii. $\begin{pmatrix}1 & 3\\ \frac{1}{2} & -2\end{pmatrix}+\begin{pmatrix}-\frac{1}{2} & 0\\ \frac{3}{2} & 1\end{pmatrix}=\begin{pmatrix}\frac{1}{2} & 3\\ 2 & -1\end{pmatrix}$

iii. $3\begin{pmatrix}1 & -1\\ 0 & 2\end{pmatrix} = \begin{pmatrix}3 & -3\\ 0 & 6\end{pmatrix}$

iv. $2\begin{pmatrix}1 & 4\\ -1 & 6\end{pmatrix} + 3\begin{pmatrix}0 & -2\\ 4 & 1\end{pmatrix} = \begin{pmatrix}2 & 14\\ 10 & 15\end{pmatrix}$

3. Calculate the following matrix products:

i. $\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}\begin{pmatrix}-1 & -2\\ 2 & 1\end{pmatrix} = \begin{pmatrix}-1 & -2\\ 0 & -3\end{pmatrix}$

ii. $\begin{pmatrix}1 & 3\\ 2 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & -3\\ 1 & 2 & -2\end{pmatrix} = \begin{pmatrix}4 & 6 & -9\\ 3 & 2 & -8\end{pmatrix}$

iii. $\begin{pmatrix}1 & \frac{1}{2} & 3\end{pmatrix}\begin{pmatrix}1 & 4\\ -1 & \frac{1}{2}\\ 0 & 1\end{pmatrix} = \begin{pmatrix}\frac{1}{2} & \frac{29}{4}\end{pmatrix}$

iv. $\begin{pmatrix}3 & -2 & 5\end{pmatrix}\begin{pmatrix}2\\ 0\\ 7\end{pmatrix} = \begin{pmatrix}41\end{pmatrix}$

11.2 Matrix properties

1. Calculate the following products:

i. $\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}\left[\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}\begin{pmatrix}-1 & -2\\ 2 & 1\end{pmatrix}\right] = \begin{pmatrix}0 & -2\\ 4 & -2\end{pmatrix}$

ii. $\left[\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}\begin{pmatrix}-1 & -2\\ 2 & 1\end{pmatrix}\right]\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix} = \begin{pmatrix}-4 & -2\\ 8 & 2\end{pmatrix}$

iii. One product was $\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}\begin{pmatrix}0 & -2\\ 4 & 2\end{pmatrix}$ and the other was $\begin{pmatrix}0 & -2\\ 4 & 2\end{pmatrix}\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}$. Notice that one is AB and the other is BA, the order has been swapped. However, the answers are different. This illustrates the general rule that if you swap the order of multiplication you get a different answer. There are special cases where the result is the same but they are rare.

2. Answer the following two questions about squaring matrices:

i. Which of the following matrices can be squared? $A = \begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}$ and $M = \begin{pmatrix}1 & 0 & -3\\ 1 & 2 & -2\end{pmatrix}$ Squaring means to multiply by itself, so we're being asked if AA and MM are allowed. Only AA meets the shape conditions.

ii. In general, considering all possible matrices, which matrices can be squared? In general, it is hopefully clear from the previous example that only matrices whose number of rows match their number of columns can be multiplied by themselves, i.e. matrices we call *square matrices*. i.e. m-by-n matrices where $m = n$.

3. For the matrix $A = \begin{pmatrix}a & b & c\\ d & e & f\end{pmatrix}$,

i. Calculate A^T and $(A^T)^T$. $A^T = \begin{pmatrix}a & d\\ b & e\\ c & f\end{pmatrix}$ and $(A^T)^T = \begin{pmatrix}a & b & c\\ d & e & f\end{pmatrix}$.

ii. How does $(A^T)^T$ compare to A? $(A^T)^T = A$ as is always the case for all matrices.

4. Let $A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$ and $B = \begin{pmatrix} u & v & w \\ x & y & z \end{pmatrix}$.

i. Evaluate $A^T + B^T$ and $(A+B)^T$. $A^T + B^T = \begin{pmatrix} a+u & d+x & \\ b+v & e & y \\ c+w & f & z \end{pmatrix}$

ii. Comment on your answer. These two calculations give the same answer. One adds before flipping the matrix, the other flips them both first then adds them. They both end up in the same place.

11.3 Matrix determinants and inverses

1. Calculate the determinants of each of the following matrices:

i. $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, Det $= 1 \times 1 - 1 \times 0 = 1$.

ii. $\begin{pmatrix} 3 & -2 \\ 4 & 5 \end{pmatrix}$, Det $= 3 \times 5 - 4 \times (-2) = 23$.

iii. $\begin{pmatrix} 6 & -3 \\ -4 & 2 \end{pmatrix}$, Det $= 6 \times 2 - (-3) \times (-4) = 0$

iv. $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$, Determinant equals... $1 \det \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - 1 \det \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + 0 \det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 2$

v. $\begin{pmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 2 & 3 & 2 \end{pmatrix}$, Determinant equals... $1 \det \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} - 2 \det \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} + 1 \det \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} = -2$

vi. $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 4 & 4 \end{pmatrix}$, Determinant equals... $2 \det \begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix} - 1 \det \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} + 1 \det \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = 0$

2. For each matrix in the previous question, determine its inverse or explain why you know it doesn't have an inverse. (You may use a computer for 3-by-3 matrix inverses)

   Matrices (c) and (f) have determinants equal to zero. That means these are the ones which cannot be inverted (i.e. have no inverse). The inverses of the others (a), (b), (d) and (e) are:

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$\frac{1}{23} \begin{pmatrix} 5 & 2 \\ -4 & 3 \end{pmatrix}$$

$$\frac{1}{2} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}$$

$$\frac{1}{2}\begin{pmatrix} -1 & 1 & 0 \\ 4 & 0 & -2 \\ -5 & -1 & 4 \end{pmatrix}$$

3. For the matrix $A = \begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix}$, use the standard formula to determine its inverse (written A^{-1}) and calculate the two products AA^{-1} and $A^{-1}A$. Comment on your results. $\det(A) = 4 \times 3 - 5 \times 2 = 12 - 10 = 2 \neq 0$ so there is an inverse. And

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 3 & -5 \\ -2 & 4 \end{pmatrix}$$

Both AA^{-1} and $A^{-1}A$ are equal to $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. A matrix multiplied by its own inverse (either way around) always equals an identity matrix, of the right shape.

4. For the matrix $A = \begin{pmatrix} 4 & 1 & 3 \\ 2 & 1 & 2 \end{pmatrix}$ calculate the product AA^T. Can you always multiply a matrix by its own transpose? $A^T = \begin{pmatrix} 4 & 2 \\ 1 & 1 \\ 3 & 2 \end{pmatrix}$ so

$$AA^T = \begin{pmatrix} 4 & 1 & 3 \\ 2 & 1 & 2 \end{pmatrix}\begin{pmatrix} 4 & 2 \\ 1 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 26 & 15 \\ 15 & 9 \end{pmatrix}$$

Yes! This is always possible, because each row length of A matches the column length in A^T because they are started life as the same rows!

5. Let $A = \begin{pmatrix} 2 & 5 \\ 1 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & -5 \\ 0 & 3 \end{pmatrix}$.

   i. First calculate A^{-1} and B^{-1}.

   $A^{-1} = \frac{1}{3}\begin{pmatrix} 4 & -5 \\ -1 & 2 \end{pmatrix}$ and $B^{-1} = \frac{1}{12}\begin{pmatrix} 3 & 5 \\ 0 & 4 \end{pmatrix}$

   ii. Next evaluate, by hand, $AB, (AB)^{-1}$ and $B^{-1}A^{-1}$

   $AB = \begin{pmatrix} 8 & 5 \\ 4 & 7 \end{pmatrix}$ and so $(AB)^{-1} = \frac{1}{36}\begin{pmatrix} 7 & -5 \\ -4 & 8 \end{pmatrix}$

   What about $B^{-1}A^{-1}$? That turns out to match $(AB)^{-1}$ exactly.

   iii. Comment on whether you were expecting these final two evaluations to be equal.

   Yes we were, because one of the always true equations for matrix inverses is as follows:

$$(AB)^{-1} = B^{-1}A^{-1}$$

6. Consider the following matrix $A = \begin{pmatrix} 1 & 2 \\ k & 3 \end{pmatrix}$ where k is a constant.

   i. Determine which value(s) of k allow A to be invertible.

      $\det(A) = 3 - 2k$, and the matrix is invertible as long as $\det(A) \neq 0$. So, as long as $k \neq \frac{3}{2}$.

   ii. Calculate the inverse of A (note your answer will contain k)

      Assuming $3 - 2k$ isn't zero, i.e. that $k \neq \frac{3}{2}$ then

$$A^{-1} = \frac{1}{3-2k} \begin{pmatrix} 3 & -2 \\ -k & 1 \end{pmatrix}.$$

7. Let $A = \begin{pmatrix} 1 & -2 & 1 \\ -3 & 2 & -1 \\ 2 & -1 & 0 \end{pmatrix}$.

   i. Calculate the determinant of A.

      $\det(A) = 2$

   ii. Find the inverse of A (using your computer).

      $A^{-1} = -\frac{1}{2} \begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 2 \\ 1 & 3 & 4 \end{pmatrix}$

   iii. By hand, calculate the matrix products AA^{-1} and $A^{-1}A$.

      Both of these will equal I_3.

   iv. Did your results agree with what you were expecting?

      Hopefully you realized that the answer was going to be I_3 before you finished the by-hand calculation. Ideally, before you even started.

8. Given that $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix}$, find the inverse matrix D^{-1} without using a computer.

$$D^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$$

   The method for this is often called *by inspection*, because it's possible to 'spot' that this is a matrix which multiplied by the original to give I_3. Though you probably won't spot it unless you've seen the pattern before!

9. (Hardest) Let $A = \begin{pmatrix} 4 & 0 & 9 \\ 0 & 5+k & -3 \\ 0 & 2 & k \end{pmatrix}$ be a matrix, where k is a constant.

   i. Calculate the determinant of A (your answer will contain k).

$$\det(A) = 4\det\begin{pmatrix} 5+k & -3 \\ 2 & k \end{pmatrix} - 0\det\begin{pmatrix} 0 & -3 \\ 0 & k \end{pmatrix} + 9\det\begin{pmatrix} 0 & 5+k \\ 0 & 2 \end{pmatrix}$$

But only the first term here isn't zero, so the answer is $\det(A) = 4\left((5+k)(k)+6\right)$ which simplifies into

$$\det(A) = 4\left(k^2+5k+6\right) = 4(k+2)(k+3).$$

ii. Use your determinant formula to determine all values of k when the matrix is invertible.

By factorizing the determinant it becomes easy to see when the matrix will be invertible, as we just need to solve to find when $\det(A) = 0$ and all the other cases will be invertible. In this case we can easily spot that $k = -2$ or $k = -3$ both cause non-invertibility. Hence all k except for -3 and -2 allow A to be inverted.

11.4 Solving simultaneous of equations

1. Consider the following system of simultaneous equations:

$$\begin{aligned} 2x - 5y &= 2 \\ 3x - 7y &= 1 \end{aligned}$$

i. Express these simultaneous equations in matrix form, i.e. as $A\underline{x} = \underline{b}$, where A is a square matrix, and both $\underline{b}$ and $\underline{x}$ are single column matrices.

$$\begin{pmatrix} 2 & -5 \\ 3 & -7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

so to fit with $A\underline{x} = \underline{b}$ we use

$$A = \begin{pmatrix} 2 & -5 \\ 3 & -7 \end{pmatrix}, \underline{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

and, as usual, we use the somewhat awkward notation of $\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ (Note that are two different x's, one the name of a column vector/matrix, $\underline{x}$, and the other the first unknown variable, x.

ii. Determine the inverse, A^{-1}, of your matrix A.

$$A^{-1} = \begin{pmatrix} -7 & 5 \\ -3 & 2 \end{pmatrix}$$

iii. Use A^{-1} to find the solution to the simultaneous equations.

The solution is always $\underline{x} = A^{-1}\underline{b}$ so

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -7 & 5 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -9 \\ -4 \end{pmatrix}$$

so $x = -9, y = -4$.

2. A system of equations is given by

$$\begin{aligned} 2x - y + 3z &= 13 \\ x - 2y - 3z &= -4 \\ 4x - 2y - 3z &= 8 \end{aligned}$$

i. Express these simultaneous equations in the matrix form

$$A\underline{x} = \underline{b}.$$

$$A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & -2 & -3 \\ 4 & -2 & -3 \end{pmatrix}$$

$$\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad \underline{b} = \begin{pmatrix} 13 \\ -4 \\ 8 \end{pmatrix}.$$

ii. Determine the matrix A^{-1} (use a computer, it will exist in this case).

$$A^{-1} = \frac{1}{27}\begin{pmatrix} 0 & -9 & 9 \\ -9 & -18 & 9 \\ 6 & 0 & -3 \end{pmatrix}.$$

iii. Use your result in the previous part to solve this system of equations for x, y, z.

As usual $\underline{x} = A^{-1}\underline{b}$ so

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{27}\begin{pmatrix} 0 & -9 & 9 \\ -9 & -18 & 9 \\ 6 & 0 & -3 \end{pmatrix}\begin{pmatrix} 13 \\ -4 \\ 8 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}.$$