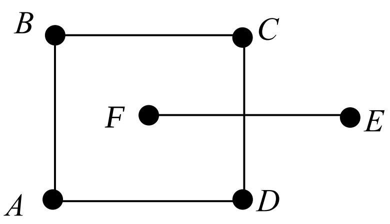
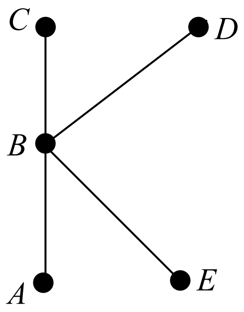
INTRODUCTION TO GRAPH THEORY - TUTORIAL

Q1. **(i).** Which of the following graphs are connected?

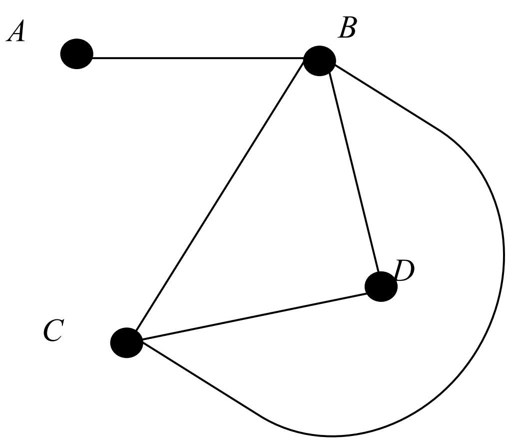
(a).



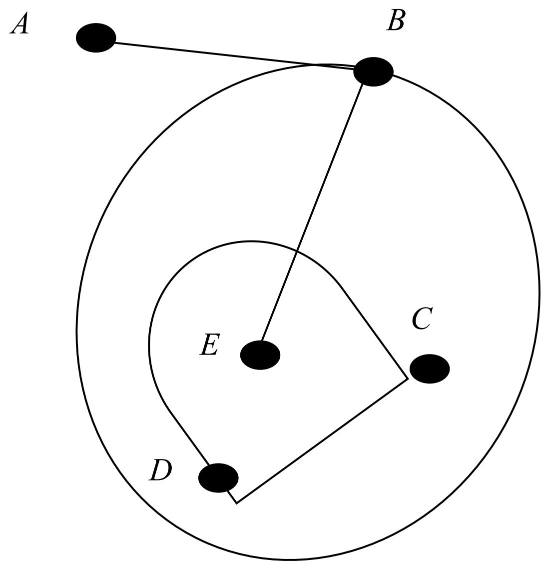
(b).



c).



(d).



(ii). If a graph is not connected state its connected components.

(iii). Which are simple graphs and which are multigraphs?

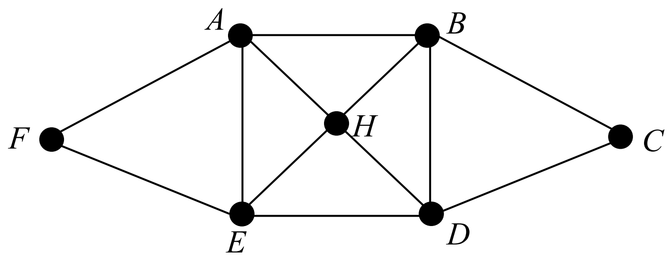
Q2. Sketch the following graphs:

(i). 4-regular on 6 vertices **(ii).** K_5 **(iii).** C_6

(iv). K_6 **(v).** $K_{2,3}$ **(vi).** $K_{4,4}$.

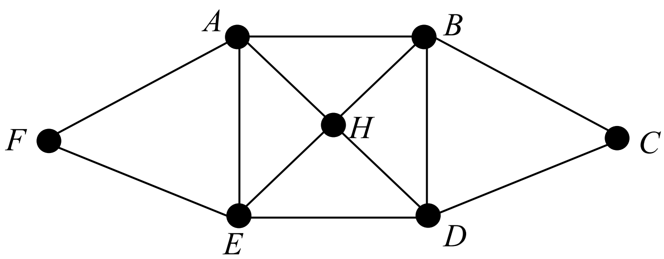
Q3. **(i).** Define the terms walk, trail and path on a graph.

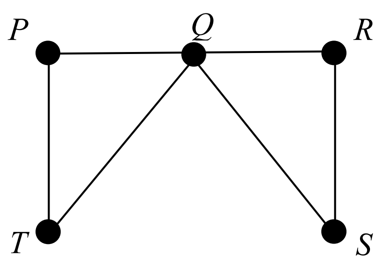
(ii). Find a walk, closed walk, trail, closed trail (circuit), path and a closed path (cycle) on the graph below.



Q4. **(i).** Define the term **Euler circuit** on a graph and find an Euler circuit on each of the graphs below if one exists. If none exist explain why not.

(ii). Define the term **Hamiltonian cycle** on a graph and find a Hamiltonian cycle on each of the graphs below if one exists. If none exist explain why not.





Q**5.** Sketch the undirected graph *G* defined below and construct an adjacency matrix for *G*.

$G = \{ V, E \} = \{ \ \{ 1, 2, 3,4, 5 \}, \ \ \{ \{1, 2\}, \{1, 3\}, \{1, 5\}, \{1, 5\},$
$\{2, 3\}, \{2, 3\}, \{3, 4\}, \{3, 5\}, \{4, 5\} \} \ \ \}.$

Q6. Consider the adjacency matrix

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}.$$

(i). Sketch the associated undirected graph, *G*, clearly labelling all the vertices.

(ii). Write down the degree sequence for *G*.

(iii). Show that the Handshaking Lemma holds for *G*.

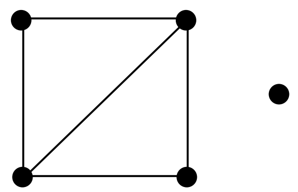
(iv). Is *G* Eulerian? Justify your answer and give an Euler circuit if appropriate.

(v). Is *G* Hamiltonian? Justify your answer and give a Hamiltonian cycle if appropriate.

(vi). Removal of an edge from *G* results in a bipartite graph. Identify which edge should be removed and sketch the resulting graph.

(vii). How many edges need to be added to *G* to obtain a complete graph? Identify which edges need to be added and sketch the resulting graph.

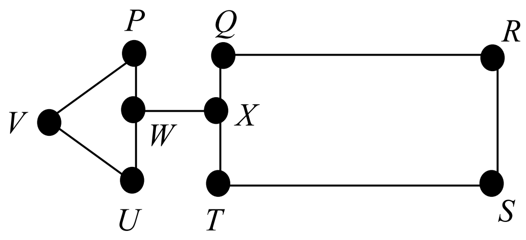
Q7. Given a graph, *G*, its **complementary graph** denoted $\bar{G}$, is obtained from *G* by replacing edges with non-edges and non-edges by edges. If *G* is given by the graph below sketch its complementary graph, $\bar{G}$.



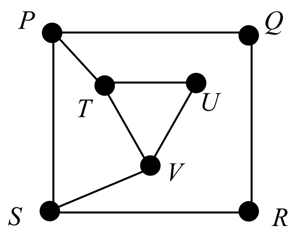
Q8. A graph, *G*, is *k*-regular if all vertices have degree *k*. Calculate the degree sum for a *k*-regular graph with *n* vertices and the number of edges in *G*.

Q9. In a simple graph, with at least two vertices, there are at least two vertices of the same degree. This result is not true for multigraphs. Sketch a three vertex multigraph with all vertices of different degree.

Q10. Consider the graph, *G* below. Explain why *G* does not have a Hamiltonian cycle.



Q11. Consider the graph, *G*, below,

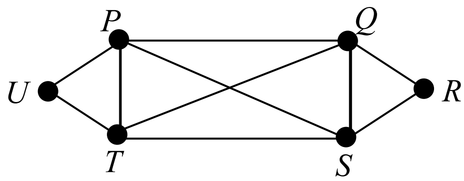


(i). Is *G* Eulerian? Either state an Euler circuit on *G* or explain why *G* is not Eulerian.

(ii). Is *G* Hamiltonian? Either state a Hamiltonian cycle on *G* or explain why *G* is not Hamiltonian.

Q12. Sketch a simple graph *G* whose vertices all have even degree but *G* is not Eulerian.

Q13. Consider the graph G below,

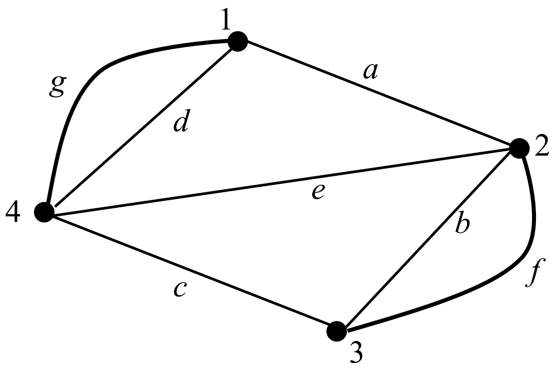


(i). Is G Eulerian? Either state an Euler circuit on G or explain why G is not Eulerian..

(ii). Is G Hamiltonian? Either state a Hamiltonian cycle on G or explain why G is not Hamiltonian.

Q14. Determine whether the complete graphs K_{77} and K_{32} are Eulerian.

Q15. Determine an adjacency matrix and an incidence matrix for the graph shown below,



Q16. An adjacency matrix for an undirected graph, *G* is given by,

$$A = \begin{pmatrix} 2 & 1 & 1 & 3 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 1 \\ 3 & 1 & 1 & 0 \end{pmatrix}.$$

Without drawing *G*, and using only the matrix *A*, answer the following:

(i). How many edges does *G* have?

(ii). How many paths of length 2 join Vertices 1 and 4.

Q17. How many edges does a tree, *T*, with 5000 vertices have?

Q18. Determine which complete bipartite graphs, $K_{m,n}$ are trees.

Q19. **(i).** Determine the conditions on *r* and *s* that will guarantee that the complete bipartite graph, $K_{r,s}$ will have an Euler circuit.

(ii). How many edges and vertices does the complete bipartite graph $K_{r,s}$ have? Give you answer in terms of *r* and *s*.

Q20. Explaining your answer state whether a graph on 7 vertices can have each vertex of degree 5.

Q21. Consider a graph *G* on 12 vertices where each vertex has degree 7. How many edges does *G* have? Explain your answer.

Q22. **(i)** Sketch the digraph

D = { { 1, 2, 3, 4 }, { {1, 2}, {1, 4}, {2, 3}, {2, 4}, {3, 2}, {3, 4}, {4, 1} } }.

(ii) Determine an adjacency matrix for *D*.

(iii). Calculate the in-degree and out-degree of each vertex.

(iv). State the Handshaking (Di)Lemma and show that it holds for *D*.

(v). State what it means for a digraph to be Eulerian.

(vi). Is the digraph, *D*, Eulerian? Explain your answer.

(vii) Calculate the matrix A^3 and explain the meaning of the entry at position (1, 2) in A^3.

Q23. Consider the following adjacency matrix, *A*, for a digraph, *D*

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

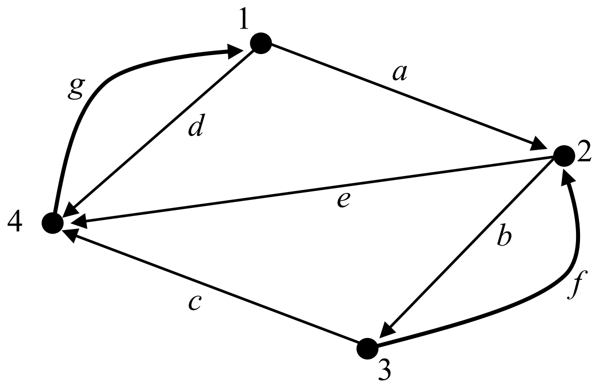
Without drawing *D*, and using only the matrix *A* answer the following:

(i). Calculate the in-degree and out-degree of each vertex.

(ii). Determine whether *D* is Eulerian. Explain your answer.

(iii). How many arcs (edges) are there in *D*? Explain your answer.

Q24. Determine an adjacency matrix for the digraph below.



Q25. Consider the following adjacency matrix

$$A = \begin{matrix} & 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & 1 & 0 & 1 & 0 \end{matrix}.$$

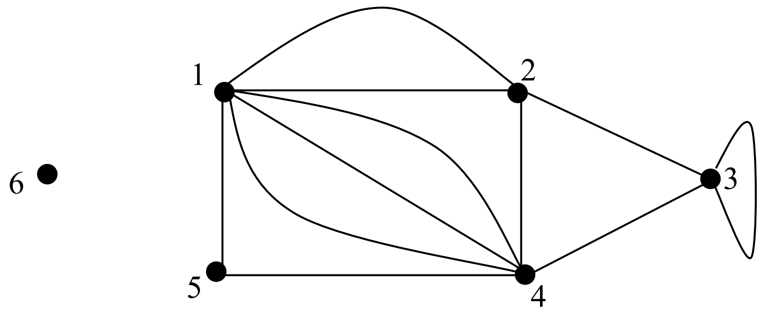
(i). Sketch the associated digraph, *D*, clearly labelling the vertices.

(ii). Determine whether the digraph is Eulerian and state an Euler circuit if one exists.

Q26. (i). In a cycle graph, C_n, state how the number of vertices is related to the number of edges.

(ii). Sketch the cycle graphs C_5 and C_6.

Q27. Consider the following multigraph, *G*.



(i). Write down an adjacency matrix for *G*. (Note that for an undirected graph we define a loop to contribute 2 to the degree of a vertex).

(ii). Interpret the row sum of the entries in row *j* of the adjacency matrix.

(iii). What is the degree of Vertex 3? Explain your answer.

Q28. **(i)** Sketch the complete graph K_5 and label the vertices *P*, *Q*, *R*, *S* and *T*.

(ii). Construct an adjacency matrix for K_5.

(iii). Describe an adjacency matrix for the general complete graph, K_n.

(iv). Interpret the row sum of the entries in row *j* of the adjacency matrix for K_n.

(v). How many 1's are contained in the adjacency matrix for a general K_n?

(vi). Interpret the result in part (v).

Q29. Let A = {1, 2, 3, 4, 5, 6} and define the relation R as follows,

R = {(1, 1), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5),
(3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4), (5, 1), (5, 2), (5, 5)} on A.

(i). Sketch the **digraph**, D, that represents R.

(ii). Determine an adjacency matrix for R.

Q30. (i). Sketch the directed (digraph) and undirected graphs corresponding to the following adjacency matrix.

$$A = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \end{matrix}.$$

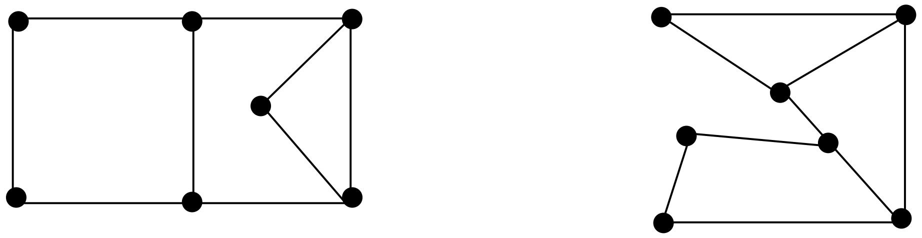
(ii). For the undirected graph determine the degree of each vertex. Then for the digraph determine the in-degree and out-degree of each vertex.

(iii). For both undirected and directed graphs determine whether they are Eulerian and/or Hamiltonian.

Q31. Explain why it is not possible to have the following adjacency matrix for a **simple** graph, (a simple graph is undirected, unweighted and has no loops or parallel edges),

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

Q32. Determine whether the two graphs below are isomorphic.



Q33. What is the chromatic number of a cycle graph, C_n?