Worked Example No.2
Deflection of Beams

For the simply supported beam shown in Figure Ex.2, determine the deflection at A and D (the mid-point of BC) and the slope at B and C. Take E = 207 GN/m^2 and I = 200x10^6 mm^4.

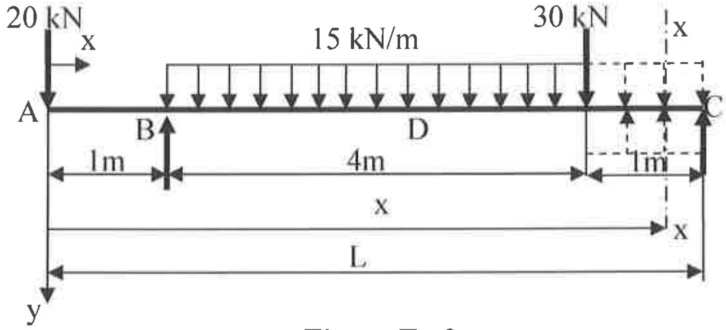


Figure Ex.2

Solution
Macauley's Method

Support reactions: $\sum M_C = 0$ (+ clockwise): $5R_B = (20x6) + (15x4x3)+(30x1) = 330$ ∴ $R_B = 66$ kN ↑

$\sum F_y = 0$ +↑ $R_B + R_C - 20 - (15x4) - 30 = 0$ ∴ $R_C = 44$ kN ↑

$$\therefore\ M_{xx} = -20x + 66(x-1) - \frac{15}{2}(x-1)^2 + \frac{15}{2}(x-5)^2 - 30(x-5)$$

$$EI\frac{d^2y}{dx^2} = -M_{xx} = 20x - 66(x-1) + \frac{15}{2}(x-1)^2 - \frac{15}{2}(x-5)^2 + 30(x-5)$$

$$EI\frac{dy}{dx} = \frac{20}{2}x^2 - \frac{66}{2}(x-1)^2 + \frac{15}{6}(x-1)^3 - \frac{15}{6}(x-5)^3 + \frac{30}{2}(x-5)^2 + A$$

$$EIy = \frac{10}{3}x^3 - \frac{33}{3}(x-1)^3 + \frac{15}{24}(x-1)^4 - \frac{15}{24}(x-5)^4 + \frac{15}{3}(x-5)^3 + Ax + B$$

Applying the boundary conditions:

i) when x = 1, y = 0: $0 = \frac{10}{3} - 0 + 0 - \text{ignored} + \text{ignored} + A + B$ ∴ $A + B = -3.333$ Eq.1

ii) when x = 6, y = 0: $0 = \frac{10}{3}(6)^3 - 11(5)^3 + \frac{5}{8}(5)^4 - \frac{5}{8}(1)^4 + 5(1)^3 + 6A + B$ ∴ $6A + B = 260$ Eq.2

From Eq.1: $B = -3.333 - A$ ∴ Eq.2 gives: $6A - 3.333 - A = 260$ ∴ $\underline{A = 52.67,\ B = -56}$

Hence, $$EI\frac{dy}{dx} = 10x^2 - 33(x-1)^2 + \frac{5}{2}(x-1)^3 - \frac{5}{2}(x-5)^3 + 15(x-5)^2 + 52.67$$

And, $$EIy = \frac{10}{3}x^3 - 11(x-1)^3 + \frac{5}{8}(x-1)^4 - \frac{5}{8}(x-5)^4 + 5(x-5)^3 + 52.67x - 56$$

These are general expressions for slope and deflection at any point 'x' along the beam. For slope and deflection at any point, the 'position' of that point 'x' is substituted into the above expressions (ignoring negative brackets!), i.e.,

For deflection at A, substitute x = 0:

$$EIy_A = 0 - \text{ignored} + \text{ignored} - \text{ignored} + \text{ignored} + 0 - 56$$

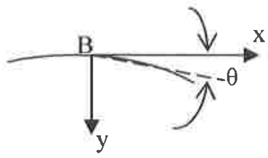
$$\therefore\ y_A = -56/EI = -56\text{x}10^3/(207\text{x}10^9 \text{ x } 200\text{x}10^{-6}) = -1.35\text{x}10^{-3}\text{ m} = \mathbf{-1.35\ mm}\uparrow$$

For deflection at D, substitute x = 3.5 m:

$$EIy_D = \frac{10}{3}(3.5)^3 - 11(2.5)^3 + \frac{5}{8}(2.5)^4 - \text{ignored} + \text{ignored} + (52.67\text{x}3.5) - 56 = 124$$

$$\therefore\ y_D = 124/EI = 124\text{x}10^3/(207\text{x}10^9 \text{ x } 200\text{x}10^{-6}) = 3\text{x}10^{-3}\text{ m} = \mathbf{3\ mm}\downarrow$$

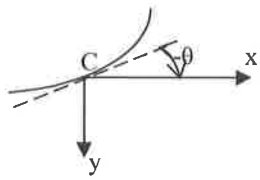
For slope at B, substitute x = 1 m: $$EI\left(\frac{dy}{dx}\right)_B = 10 - 0 + 0 - \text{ignored} + \text{ignored} + 52.67 = 62.67$$



$$\therefore\ \left(\frac{dy}{dx}\right)_B = 62.67\text{x}10^3/(207\text{x}10^9 \text{ x } 200\text{x}10^{-6}) = \mathbf{0.00152\ rad = 0.0871°}$$

For slope at C, substitute x = 6 m:

$$EI\left(\frac{dy}{dx}\right)_C = 10(6)^2 - 33(5)^2 + \frac{5}{2}(5)^3 - \frac{5}{2}(1)^3 + 15(1)^2 + 52.67 = -86.67$$



$$\therefore\ \left(\frac{dy}{dx}\right)_C = -86.67\text{x}10^3/(207\text{x}10^9 \text{ x } 200\text{x}10^{-6}) = \mathbf{-0.0021\ rad = 0.12°}$$

Note: To find the value and position of the maximum deflection y_{max}, the expression for $y = \frac{1}{EI}[f(x)]$ can be used if the position 'x' is known. In order to determine this position 'x', the condition that dy/dx=0 gives either a maximum or minimum for the function y=f(x). Hence if the equation for slope (dy/dx) is known, it is then quite simple to equate the LHS to zero and solve for 'x', whereupon this value can be substituted into the EIy expression to obtain y_{max}.