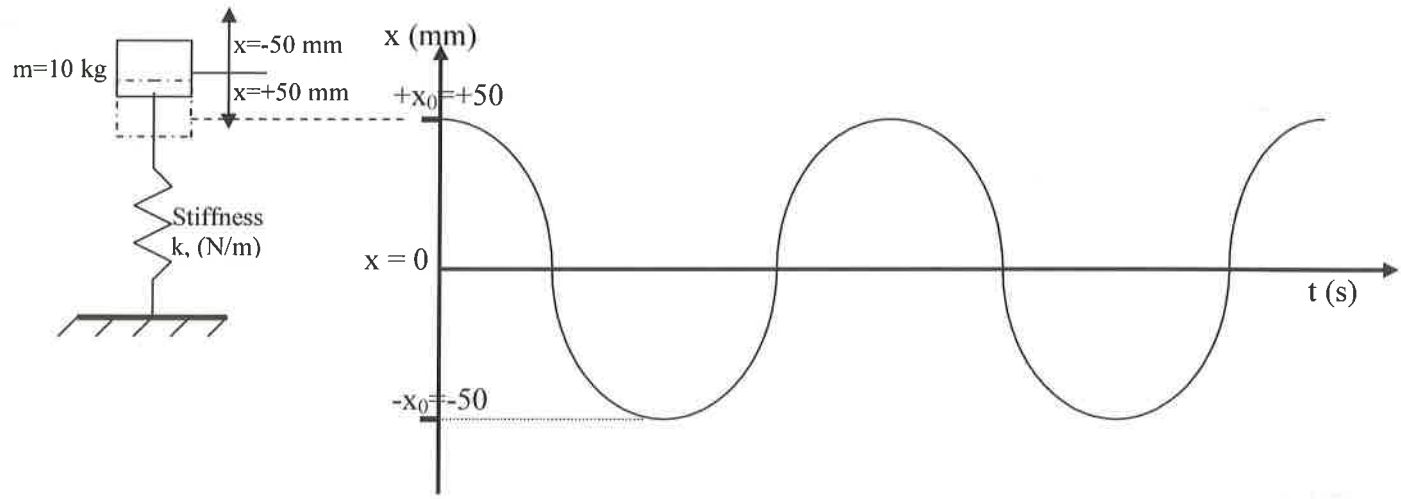
Worked Example No.1
Damped Free Vibration

A body of mass 10 kg is lowered gradually on to a spring producing a static deflection of the spring of 50 mm. The body is then displaced and released.

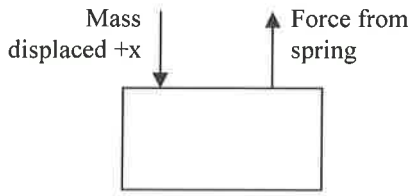
(a) Derive an equation of motion for the body.
(b) Determine the period of undamped oscillation of the body.
(c) Determine the new period of oscillation of the body by replacing the single spring with two similar springs arranged in series.
(d) Explain the effect of the vibration of the body given that damping of 150 Ns/m is applied to the system and determine the value of the critical damping coefficient for the system with the two springs.

Solution

(a)



Free body diagram for the mass in its displaced position:



$$\Sigma F = ma \quad \text{i.e.} \quad ma = m.\frac{d^2x}{dt^2} = m\ddot{x} = -kx$$

$$\therefore \boxed{m\ddot{x} + kx = 0} \longrightarrow \underline{\text{Equation of motion}}$$

The solution is $x = x_0\cos\omega_n t$ where ω is the frequency of vibration in rad/s, and $\omega_n = \sqrt{\frac{k_T}{m}}$ where k_T = total system stiffness (N/m) and m = mass (kg).

(b)

$$\omega_n = \sqrt{\frac{k_T}{m}} \qquad \text{where } k_T = \frac{F}{\delta} = \frac{10x9.81}{0.05} = 1962N/m, \text{ and m = 10 kg}$$

$$\therefore \qquad \omega_n = \sqrt{\frac{1962}{10}} = 14rad/s$$

$$\therefore \qquad f_n = \frac{\omega_n}{2\pi} = \frac{14}{2\pi} = 2.229Hz$$

The periodic time (i.e. the time for one complete oscillation): $t = \frac{1}{f_n} = \frac{1}{2.229} = 0.449s$

(c)

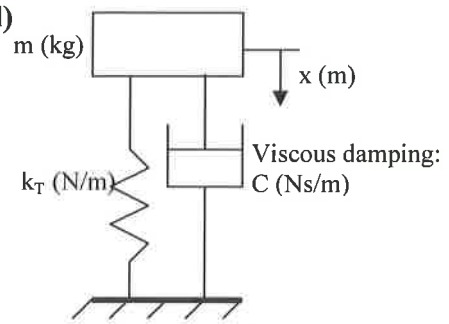
For 2 springs in series, $\frac{1}{k_T} = \frac{1}{k_1} + \frac{1}{k_2} = \frac{1}{1962} + \frac{1}{1962} = 0.00102m/N \quad \therefore \quad k_T = 981$ N/m

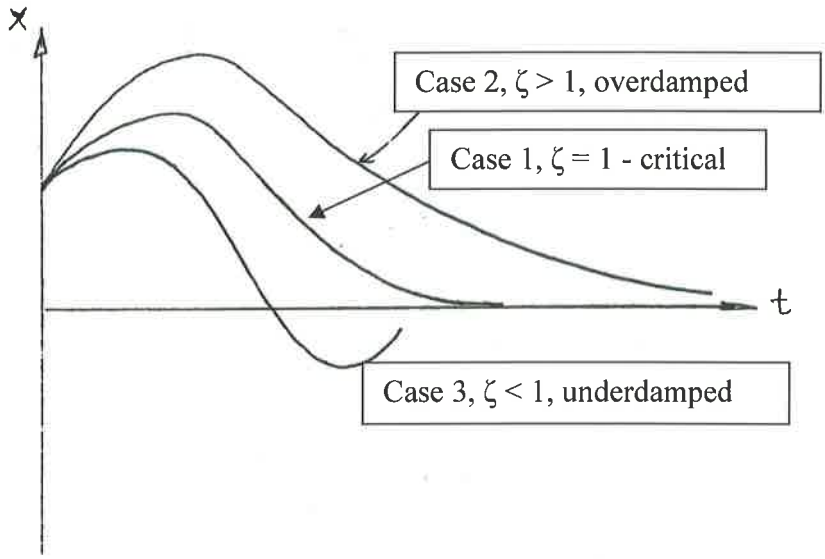
$$\therefore \qquad \omega_n = \sqrt{\frac{k_T}{m}} = \sqrt{\frac{981}{10}} = 9.904rad/s$$

$$\therefore \qquad f_n = \frac{\omega_n}{2\pi} = \frac{9.904}{2\pi} = 1.576Hz$$

$$\therefore \qquad t = \frac{1}{f_n} = \frac{1}{1.576} = 0.634s$$

(d)





Damping ratio, $\zeta = \frac{C}{C_c} = \frac{150}{198.1} = 0.757$

where $C_c = 2m\omega_n = 2x10x9.904$
$= \mathbf{198.1\ Ns/m}$

and, $C = 150$ Ns/m

Since $\zeta < 1$ (underdamped), then body oscillation will decay over time.

Worked Example No.2
Damped Free Vibration

A motor of mass 20 kg is mounted on a base/platform system with 4 springs connected in parallel with each spring having a stiffness of 10 kN/m. Viscous damping is also present in the system. The motor is displaced 10 mm in a vertical direction from its equilibrium position and is allowed to vibrate freely. After 3 cycles the vibration amplitude is reduced to 5 mm. Determine:

i) the natural frequency of the system;
ii) the viscous damping ratio;
iii) the damped frequency;
iv) the viscous damping coefficient.

Solution

i) For 4 springs arranged in parallel:

$k_T = k_1 + k_2 + k_3 + k_4 = 4(10x10^3) = 40 \text{ x } 10^3$ N/m

$$\omega_n = \sqrt{\frac{k_T}{m}} = \sqrt{\frac{40x10^3}{20}} = \underline{44.72 rad/s}$$

ii) $\delta = \zeta\omega_n\tau$ where $\delta = \frac{1}{n}\ln\frac{x_1}{x_2}$ where x_1 = 10 mm, and at n = 3 cycles, x_3 = 5 mm

$$\therefore \quad \delta = \frac{1}{3}\ln\frac{10}{5} = 0.231$$

and $$\tau = \frac{1}{f_n} = \frac{2\pi}{\omega_n} = \frac{2\pi}{44.72} = 0.1405s$$

$$\therefore \quad \zeta = \frac{\delta}{\omega_n\tau} = \frac{0.231}{44.72x0.1405} = \underline{0.037}$$ (<1, underdamped!)

iii) $\omega_d = \omega_n\sqrt{1-\zeta^2} = 44.72x\sqrt{1-0.037^2} = \underline{44.69 rad/s}$ (very close to natural frequency – resonance!)

iv) $C_c = 2m\omega_n = 2 \text{ x } 20 \text{ x } 44.72 = 1788.8$ Ns/m

$$\zeta = \frac{C}{C_c} \qquad C = \zeta C_c = 0.037x1788.8 = \underline{66.19 Ns/m}$$