



**School of Computing, Engineering &
Built Environment**

Mathematics Summer School

Level 3 Entry – Engineering

Ordinary Differential Equations

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1. Introduction

A differential equation is an equation that contains an unknown function, which we need to solve for, and its derivatives. For example,

$$\frac{dy}{dx} = x^2 - 1$$

$$\frac{dy}{dx} - y = 0$$

$$a \frac{d^2x}{dt^2} + b \frac{dx}{dt} + c x = 0$$

are all differential equations. Technically they are **ordinary** differential equations (ODEs) since they contain ordinary derivatives as opposed to partial derivatives. An equation that contains partial derivatives is called a **partial** differential equation (PDE). The equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

is an example of a partial differential equation. In this module we shall only consider **ordinary differential equations**. Differential equations are extremely important in science and engineering as they can mathematically describe physical processes such as current flow in electrical systems, motion of mechanical systems, fluid flow, chemical reactions, population dynamics, the spread of infectious diseases, and many other natural phenomena.

2. Ordinary Differential Equations – The Basics

Given an ODE in terms of $\frac{dy}{dx}$, where y is called the **dependent** variable and x is the **independent** variable, our aim is to solve the ODE and determine the function, or functions, that satisfy the equation.

Consider the simple ODE

$$\frac{dy}{dx} = x^2 - 1 . \quad [1]$$

The solution we desire will be of the form

$$y = f(x) .$$

In this case the task is straightforward as we seek the function, y , whose derivative is,

$$x^2 - 1 .$$

Hence,

$$y = \int (x^2 - 1) dx \text{ and so}$$

$$\underline{\underline{y = \frac{1}{3}x^3 - x + C .}} \quad [2]$$

Note that an arbitrary constant (C) has been generated. This means that expression [2] does not represent just a single solution of ODE [1], but infinitely many (one for each possible value of C). Expression [2] is called the **general solution** (GS) of [1] since it covers **all** possible solutions of the given ODE.

The **order** of an ODE is given by the highest derivative appearing in it. For example,

$$\frac{dy}{dx} = x^2 - 1 \quad 1^{\text{st}} \text{ order ODE}$$

$$a \frac{d^2x}{dt^2} + b \frac{dx}{dt} + c x = 0 \quad 2^{\text{nd}} \text{ order ODE}$$

$$y \frac{d^2y}{dx^2} = \left(\frac{dy}{dx} \right)^3 \quad 2^{\text{nd}} \text{ order ODE.}$$

The order of an ODE determines the number of arbitrary constants appearing in its general solution:

- 1st order - 1 arbitrary constant
- 2nd order - 2 arbitrary constants.

We shall not be considering ODEs of order higher than 2.

Once the general solution of an ODE has been found, a **particular solution** may be determined by applying **boundary** or **initial conditions**. For the simple first order ODE [1] above we might be asked to solve

$$\frac{dy}{dx} = x^2 - 1, \text{ subject to } y(0) = 1 .$$

That is, from the infinity of solutions of the ODE we are looking for the one that will give $y = 1$ when $x = 0$. The general solution was found earlier to be

$$y = \frac{1}{3}x^3 - x + C .$$

To determine the particular solution that satisfies the given condition, we determine a value for C by substituting $y = 1$ and $x = 0$ into the general solution and solving the resulting equation, i.e.

$$1 = 0 - 0 + C$$

so that

$$C = 1 ,$$

giving

$$\underline{\underline{y = \frac{1}{3}x^3 - x + 1}}$$

which satisfies **both** the ODE **and** the condition.

In general, we require one condition per arbitrary constant to determine a particular solution. Hence, the number of conditions required equals the order of the ODE.

In the simple case above we were able to determine the general solution of ODE [1] by a single, **direct integration**. Other ODEs may require different solution methods. We shall now consider some special types of ODEs and how to solve them.

3. First Order Separable ODEs

An ODE is **separable** if it can be written in the form

$$f(y) \frac{dy}{dx} = g(x) .$$

Integrating both sides, with respect to x , we obtain

$$\int f(y) \frac{dy}{dx} dx = \int g(x) dx$$

or

$$\int f(y) dy = \int g(x) dx .$$

Provided we can perform the two integrations we can obtain a solution to the ODE.

Examples / Over the page . . .

Examples

In the following examples, Examples (1) – (5) illustrate the basic processes involved when solving separable ODEs. Example (6) illustrates how an initial condition can be applied as soon as the constant of integration appears. Example (7) shows that some solutions cannot be manipulated into the **explicit** form $y = f(x)$, and so we have to be content with an **implicit** form of solution.

E1. ODE: $\frac{dy}{dx} = x^2 + 1$

Separate: $dy = (x^2 + 1) dx$

Form integrals: $\int 1 dy = \int (x^2 + 1) dx$

Integrate: $y = \underline{\underline{\frac{1}{3}x^3 + x + C}}$. (General solution)

[Note that we only need **one** constant of integration]

E2. ODE: $\frac{dy}{dx} = \frac{x}{y}$

Separate: $y dy = x dx$

Form integrals: $\int y dy = \int x dx$

Integrate: $\frac{1}{2} y^2 = \frac{1}{2} x^2 + C$

Tidy up: $y^2 = x^2 + K$, ($K = 2C$)

General soln: $\underline{\underline{y = \pm\sqrt{x^2 + K}}}$.

E3. ODE: $\frac{dy}{dx} = y(1 - 3x^2)$

Separate: $\frac{dy}{y} = (1 - 3x^2) dx$

Form integrals: $\int \frac{dy}{y} = \int (1 - 3x^2) dx$

Integrate: $\ln|y| = x - x^3 + C$

Tidy up: $y = e^{x-x^3+C}$

$$y = e^{x-x^3} e^C$$

General soln: $\underline{\underline{y = K e^{x-x^3}}}$, $(K = e^C)$.

E4. ODE: $\frac{dy}{dx} = e^{x-y}$

Re-arrange: $\frac{dy}{dx} = \frac{e^x}{e^y}$

Separate: $e^y dy = e^x dx$

Form integrals: $\int e^y dy = \int e^x dx$

Integrate: $e^y = e^x + C$

General soln: $\underline{\underline{y = \ln|e^x + C|}}$.

E5. ODE: $\frac{dy}{dx} = (y^2 + 1)(3x^2 + 4)$

Rearrange: $\frac{1}{(y^2 + 1)} \frac{dy}{dx} = (3x^2 + 4)$

Separate: $\frac{dy}{(y^2 + 1)} = (3x^2 + 4)dx$

Form integrals: $\int \frac{dy}{(y^2 + 1)} = \int (3x^2 + 4) dx$

Integrate: $\tan^{-1}(y) = x^3 + 4x + C$

General soln: $y = \tan(x^3 + 4x + C)$.

E6. ODE + condition: $\frac{dy}{dx} = y(1 - 3x^2)$, $y(2) = 1$

From Example 3: $\ln|y| = x - x^3 + C$

Apply condition: $y = 1$ when $x = 2$

$$\ln|1| = 2 - 2^3 + C$$

$$0 = -6 + C$$

$$C = 6 .$$

Substitute value: $\ln|y| = x - x^3 + 6 .$

Particular Soln: $y = e^{x - x^3 + 6}$.

E7. ODE + condition: $\frac{dy}{dx} = \frac{2x + e^x}{1 + e^y}$, $y(2) = 0$

Separate: $(1 + e^y) dy = (2x + e^x) dx$

Form integrals: $\int (1 + e^y) dy = \int (2x + e^x) dx$

Integrate: $y + e^y = x^2 + e^x + C$

Apply condition: $y = 0$ when $x = 2$

$$0 + e^0 = 2^2 + e^2 + C$$

$$1 = 4 + e^2 + C$$

$$C = -(3 + e^2)$$

Particular Soln: $y + e^y = x^2 + e^x - (3 + e^2)$

[Note that as we cannot solve explicitly for y the solution is left in **implicit** form]

Now watch the following:

<https://www.youtube.com/watch?v=nlvr3UyMiQ4>

<https://www.youtube.com/watch?v=TojF0AAOdW0>

<https://www.youtube.com/watch?v=M54Ymxf7ATc>

4. Linear ODEs

An ODE is **linear** if it is linear in the dependent variable and its derivatives. No multiplications involving the dependent variable and its derivatives are allowed. In a linear ODE all coefficients must therefore be constants or functions of the independent variable. An equation that is not linear is **nonlinear**. Some examples of linear and nonlinear ODEs are :

$$x^2 \frac{dy}{dx} + y^3 = \cos(x) \quad 1^{\text{st}} \text{ order nonlinear} \quad (y^3 \text{ is the nonlinear term})$$

$$x \frac{dy}{dx} - y = x^2 e^{3x} \quad 1^{\text{st}} \text{ order linear}$$

$$\frac{d^2 y}{dx^2} + y \frac{dy}{dx} + y = 0 \quad 2^{\text{nd}} \text{ order nonlinear} \quad (y \frac{dy}{dx} \text{ is the nonlinear term})$$

$$3 \frac{d^2 y}{dx^2} + 10 \frac{dy}{dx} - 8y = \sin(2x) \quad 2^{\text{nd}} \text{ order linear}$$

$$e^{3x} \frac{dy}{dx} + \sin(y) = 5x^2 \quad 1^{\text{st}} \text{ order nonlinear} \quad (\sin(y) \text{ is the nonlinear term}).$$

Before we look at the next method for solving first order ODEs it will be helpful to recall some useful results:

Laws of Logarithms

- $N \ln(M) = \ln(M^N)$
- $\ln(MN) = \ln(M) + \ln(N)$
- $\ln\left(\frac{M}{N}\right) = \ln(M) - \ln(N)$
- $e^{\ln(A)} = A$

Integration by Parts

$$\int f(x)g'(x) dx = f(x)g(x) - \int f'(x)g(x) dx.$$

5. First Order Linear ODEs - The Integrating Factor Method

Consider an ODE that can be written in the **standard form**

$$\frac{dy}{dx} + p(x) y = q(x) \quad [3]$$

where $p(x)$ and $q(x)$ are non-zero and not equal. This type of equation is known as a **first order linear ODE** and can be solved using the **integrating factor method**. The solution procedure involves the following steps:

1. Write the ODE in standard form.
2. Identify $p(x)$ and determine the integrating factor (IF), $r(x) = e^{\int p(x) dx}$.
3. Multiply the standard form ODE [3] by $r(x)$:

$$r(x) \frac{dy}{dx} + r(x) p(x) y = r(x) q(x).$$

4. Rewrite as, $\frac{d}{dx} [r(x) y] = r(x) q(x)$.
5. Integrate both sides, with respect to x , to obtain the general solution:

$$r(x) y = \int r(x) q(x) dx \quad \Rightarrow \quad y = \frac{1}{r(x)} \int r(x) q(x) dx.$$

Note that we can actually omit Steps 3 and 4 and, after calculating the integrating factor at Step 2, write the general solution (Step 5) as, $y = \frac{1}{r(x)} \int r(x) q(x) dx$.

Solving ODEs of the form [3] using integrating factors means performing two integrations, one to determine the IF and one to eliminate the derivative. It might appear that we end up with two constants of integration when we only want one. However, it is easily demonstrated that any constant arising from calculating $r(x)$ eventually cancels out. In practice, we simply set it to zero as soon as it arises leaving just one arbitrary constant from the final integration. As before, if an initial condition is given then the constant of integration can be evaluated once it appears.

Examples

E8. Determine the general solution of the ODE, $\frac{dy}{dx} + y = x$.

S1. The ODE is already in standard form.

S2. Identify $p(x)$ and determine the integrating factor:

$$p(x) = 1, \quad r(x) = e^{\int p(x) dx} = e^{\int 1 dx} = e^x.$$

S3. Write down the general solution noting that $q(x) = x$:

$$y = \frac{1}{r(x)} \int r(x) q(x) dx$$
$$y = \frac{1}{e^x} \int e^x x dx . \quad (*)$$

Use integration by parts to calculate, $\int x e^x dx$.

The formula for integration by parts gives:

$$\int f(x) g'(x) dx = f(x) g(x) - \int f'(x) g(x) dx$$

where

$$f(x) = x \quad g'(x) = e^x$$
$$f'(x) = 1 \quad g(x) = e^x.$$

Hence,

$$\int x e^x dx = x e^x - \int 1 \cdot e^x dx$$
$$= x e^x - \int e^x dx$$
$$= x e^x - e^x + C.$$

Using (*) we can now write down the general solution of the ODE:

$$y = \frac{1}{e^x} \int e^x x \, dx = \frac{1}{e^x} [x e^x - e^x + C]$$

$$\underline{\underline{y = x - 1 + \frac{C}{e^x} .}}$$

E9. Determine the general solution of the ODE, $2x \frac{dy}{dx} + y = x^2$.

S1. Write the ODE in standard form by dividing through by the coefficient of $\frac{dy}{dx}$:

$$\frac{dy}{dx} + \frac{1}{2x} y = \frac{1}{2} x .$$

S2. Identify $p(x) = \frac{1}{2x}$ and determine the integrating factor:

$$\begin{aligned} r(x) &= e^{\int p(x) \, dx} \\ &= e^{\int \frac{1}{2x} \, dx} \\ &= e^{\frac{1}{2} \int \frac{1}{x} \, dx} \\ &= e^{\frac{1}{2} \ln|x|} = e^{\ln|x^{1/2}|} = x^{1/2} \end{aligned}$$

S3. Write down the general solution, noting that $q(x) = \frac{1}{2} x$:

$$y = \frac{1}{x^{1/2}} \int x^{1/2} \frac{1}{2} x \, dx$$

$$y = \frac{1}{2} x^{-1/2} \int x^{3/2} \, dx .$$

$$y = \frac{1}{2} x^{-1/2} \left(\frac{2}{5} x^{5/2} + C \right) \Rightarrow \underline{\underline{y = \frac{1}{5} x^2 + \frac{K}{x^{1/2}} , \quad (K = C/2) .}}$$

Now watch the following:

<https://www.youtube.com/watch?v=GhGn21sKong>

<https://www.youtube.com/watch?v=cdrhTsSOkcU>

6. Second Order, Linear ODEs with Constant Coefficients

In this section we look at special types of second order ODEs. The following definitions will prove useful for the understanding of parts of this section.

Definition (1): Given two functions $y_1(x)$ and $y_2(x)$ a **linear combination** of these functions takes the form

$$y(x) = a_1 y_1(x) + a_2 y_2(x)$$

where a_1 and a_2 are constants.

Definition (2): Two functions, $y_1(x)$ and $y_2(x)$ are said to be **linearly independent** if one function is **not** a constant multiple of the other, i.e.

$$y_1(x) \neq k y_2(x) .$$

Definition (3): If two functions, $y_1(x)$ and $y_2(x)$ are both solutions of a linear homogeneous ODE then

$$y(x) = A y_1(x) + B y_2(x),$$

where A and B are constants, is also a solution of the ODE.

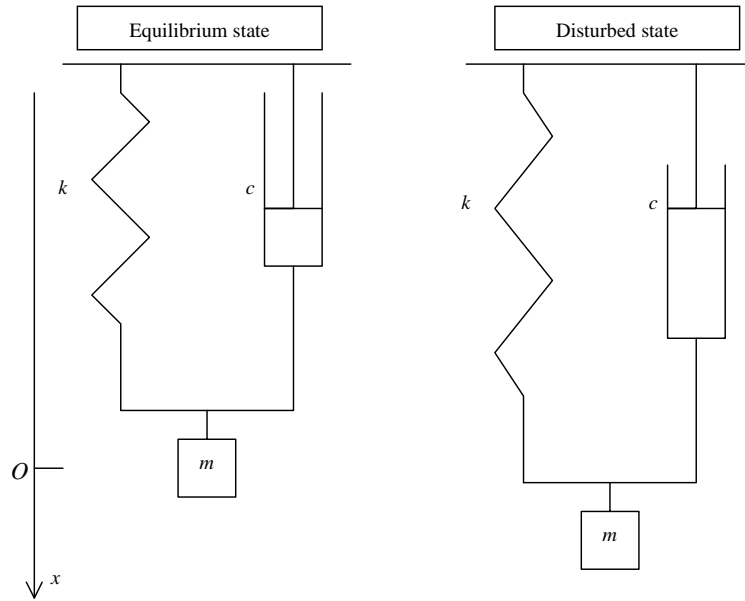
Linear ODEs with constant coefficients come in two forms:

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = 0 \quad \textbf{(Homogeneous)} \quad [4]$$

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = f(x) \quad \textbf{(Non-homogeneous)} \quad [5]$$

where a , b and c are constants.

Equations of the above form are extremely useful for modelling certain types of physical systems. Consider a simple suspension system linking a mass to some fixture via a spring and dashpot:



In this system:

m = mass

k = spring stiffness

c = damping coefficient

x = displacement of mass

$\frac{dx}{dt}$ = velocity of mass

$\frac{d^2x}{dt^2}$ = acceleration of mass

Force on mass due to spring = $-k x$

Force on mass due to dashpot = $-c \frac{dx}{dt}$.

Newton's 2nd law of motion states that $Mass \times Acceleration = Force$, so

$$m \frac{d^2 x}{dt^2} = -k x - c \frac{dx}{dt}$$

or

$$m \frac{d^2 x}{dt^2} + c \frac{dx}{dt} + k x = 0 . \quad (\text{c.f. Equation [4]})$$

If there are other forces acting on the mass (such as a forced vibration) then we obtain

$$m \frac{d^2 x}{dt^2} + c \frac{dx}{dt} + k x = f(t) \quad (\text{c.f. Equation [5]})$$

where $f(t)$ is called the forcing term.

Similar ODEs can be derived that model the behaviour of current or voltage in electronic circuits. These type of equations are applicable to many areas where oscillations or vibrations are present.

6.1. Homogeneous Equations

Consider the second order, linear, homogeneous ODE with constant coefficients

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = 0. \quad [6]$$

To obtain a solution (y) of this type of ODE we first note that in order to satisfy [6] some linear combination of the function y and its derivatives must equal zero. One class of function for which this property holds is the exponential function as differentiation corresponds to multiplication of the previous derivative by some constant.

We shall therefore consider a solution y of the form

$$y = e^{\lambda x}$$

where λ is a constant. If we differentiate this function twice we obtain

$$\frac{dy}{dx} = \lambda e^{\lambda x}$$

and

$$\frac{d^2 y}{dx^2} = \lambda^2 e^{\lambda x} .$$

Note that y and its derivatives are constant multiples of each other and have a common factor of $e^{\lambda x}$. If we substitute these expressions into ODE [6] we obtain

$$(a \lambda^2 + b \lambda + c) e^{\lambda x} = 0$$

or

$$a \lambda^2 + b \lambda + c = 0 , \quad [7]$$

since $e^{\lambda x} \neq 0$. If we choose λ to satisfy this quadratic equation then, automatically, the function $y = e^{\lambda x}$ will be a particular solution of the ODE.

Equation [7] is called the **auxiliary equation** (AE), or **characteristic equation**, of the ODE. Solving it allows us to construct particular solutions and, hence, the general solution of the ODE. Since we have to solve a quadratic equation we have to deal separately with the different types of solution that can occur. Recall that the quadratic formula gives the solution of the AE as

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} .$$

and so we must consider different cases depending on the value of the discriminant, $b^2 - 4ac$.

Case (i) $b^2 - 4ac > 0$

If $b^2 - 4ac > 0$ we obtain **two real and distinct** roots of the AE, i.e.

$$\lambda = \lambda_1 \quad \text{and} \quad \lambda = \lambda_2 .$$

Hence,

$$y_1 = e^{\lambda_1 x}$$

will be **one** solution of the ODE and

$$y_2 = e^{\lambda_2 x}$$

will be **another**. These solutions are linearly independent and by Definition (3) above any linear combination of them will also be a solution of the ODE, i.e.

$$y = A e^{\lambda_1 x} + B e^{\lambda_2 x} \quad [8]$$

will be a solution. In fact, expression [8] covers **all** possible solutions of the ODE (when $b^2 - 4ac > 0$) and is therefore the **general solution** (GS) of the ODE. Note the **two** arbitrary constants. We have constructed the general solution without the need to integrate.

Case (ii) $b^2 - 4ac < 0$

In this case we obtain **two complex conjugate** roots of the auxiliary equation, i.e.

$$\lambda_1 = \alpha + j\beta \quad \text{and} \quad \lambda_2 = \alpha - j\beta .$$

Just as in case (i) the general solution can be written down as

$$y = \tilde{A} e^{\lambda_1 x} + \tilde{B} e^{\lambda_2 x}$$

or

$$y = \tilde{A} e^{(\alpha + j\beta)x} + \tilde{B} e^{(\alpha - j\beta)x}$$

or

$$y = \tilde{A} e^{\alpha x} e^{j\beta x} + \tilde{B} e^{\alpha x} e^{-j\beta x}$$

or

$$y = e^{\alpha x} \left[\tilde{A} e^{j\beta x} + \tilde{B} e^{-j\beta x} \right] .$$

Using Euler's formula we can write

$$e^{j\beta x} = \cos(\beta x) + j \sin(\beta x)$$

and

$$e^{-j\beta x} = \cos(\beta x) - j \sin(\beta x) .$$

Hence, the general solution of the ODE can be rewritten as

$$y = e^{\alpha x} \left[\tilde{A} [\cos(\beta x) + j \sin(\beta x)] + \tilde{B} [\cos(\beta x) - j \sin(\beta x)] \right]$$

or

$$y = e^{\alpha x} [A \cos(\beta x) + B \sin(\beta x)] \quad [9]$$

where $A = (\tilde{A} + \tilde{B})$ and $B = j(\tilde{A} - \tilde{B})$.

Equation [9] provides us with a template solution when the auxiliary equation yields complex conjugate roots.

Case (iii) $b^2 - 4ac = 0$

In this case the roots of the auxiliary equation are **real and equal**, i.e.

$$\lambda = \lambda_0 .$$

Both roots give the same solution

$$y = e^{\lambda_0 x} .$$

However, we need to find two linearly independent solutions to the ODE in order to determine the general solution.

In this special case it can be shown that

$$y = e^{\lambda_0 x} \tag{10}$$

and

$$y = x e^{\lambda_0 x} \tag{11}$$

are both linearly independent solutions of the ODE and so taking a linear combination of [10] and [11] gives us the general solution

$$y = A e^{\lambda_0 x} + B x e^{\lambda_0 x}$$

or

$$y = (A + B x) e^{\lambda_0 x} .$$

Now watch the following: https://www.youtube.com/watch?v=cszjWPRqP_Q

Summary

To solve the ODE: $a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + c y = 0$:

(I). form the auxiliary equation $a \lambda^2 + b \lambda + c = 0$;

(II). solve the auxiliary equation;

(III). write down the general solution of the ODE, depending on the nature of the solution of the auxiliary equation:

Two real distinct roots: $y = A e^{\lambda_1 x} + B e^{\lambda_2 x}$

Two complex roots: $y = e^{\alpha x} [A \cos(\beta x) + B \sin(\beta x)]$

Two real equal roots: $y = (A + B x) e^{\lambda_0 x}$.

Examples

E10. Determine the general solution of the ODE, $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$.

Auxiliary equation:

$$\lambda^2 - \lambda - 2 = 0$$

$$(\lambda - 2)(\lambda + 1) = 0$$

$$\lambda = 2 \quad , \quad \lambda = -1$$

Two real and distinct roots so the general solution is: $y = A e^{2x} + B e^{-x}$.

Now watch the following: <https://www.youtube.com/watch?v=4PGYVPUUpCFc> (Real distinct roots)

E11. Determine the general solution of the ODE, $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$

Auxiliary equation:

$$\lambda^2 + \lambda + 1 = 0$$

$$\lambda = \frac{-1 \pm \sqrt{-3}}{2}$$

$$= -\frac{1}{2} \pm j \frac{\sqrt{3}}{2}, \quad \alpha = -\frac{1}{2}, \quad \beta = \frac{\sqrt{3}}{2}$$

Complex roots, so the general solution is:

$$\underline{\underline{y = e^{-\frac{1}{2}x} \left[A \cos\left(\frac{\sqrt{3}}{2}x\right) + B \sin\left(\frac{\sqrt{3}}{2}x\right) \right]}}$$

Now watch the following: <https://www.youtube.com/watch?v=2ig5MG5IacY> (Complex roots)

E12. Determine the general solution of the ODE, $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$.

Auxiliary equation:

$$\lambda^2 + 2\lambda + 1 = 0$$

$$(\lambda + 1)(\lambda + 1) = 0$$

$$\lambda = -1 \text{ (twice).}$$

Equal (repeated) roots, so the general solution is: $\underline{\underline{y = (A + Bx)e^{-x}}}$.

Now watch the following: <https://www.youtube.com/watch?v=LOZcexdamUY> (Repeated roots)

6.1.1. Homogeneous ODEs with Initial Conditions

Since the general solution of a 2nd order ODE has **two** arbitrary constants we require **two** initial conditions to obtain a particular solution. These conditions may be given in the form

$$y(a) = c_0 \qquad \frac{dy}{dx}(a) = c_1$$

where the second condition is applied to the **derivative** of the general solution. In general, we will obtain two simultaneous equations which are solved for A and B to completely solve the ODE. This type of problem, when all the constraints are specified at the same value of the independent variable (x), is known as an **initial value problem (IVP)**.

E13. Consider the spring-mass-dashpot system at the start of the section. Set $m = 1$, $c = 1$ and $k = 1$. Determine the motion of the mass if it is initially displaced by an amount h and then released from rest.

ODE:
$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + x = 0$$

Initial displacement: $x(0) = h$

Initial velocity: $\frac{dx}{dt}(0) = 0$

The ODE is just that from Example 11 with y and x replaced by x and t respectively.

Hence, the GS is

$$\underline{\underline{x = e^{-\frac{1}{2}t} \left[A \cos\left(\frac{\sqrt{3}}{2}t\right) + B \sin\left(\frac{\sqrt{3}}{2}t\right) \right] .}}$$

Apply first condition, $x(0) = h$:

$$h = e^{-\frac{1}{2} \times 0} \left[A \cos\left(\frac{\sqrt{3}}{2} \cdot 0\right) + B \sin\left(\frac{\sqrt{3}}{2} \cdot 0\right) \right]$$

$$A = h .$$

Substitute for A in the GS

$$x = e^{-\frac{1}{2}t} \left[h \cos\left(\frac{\sqrt{3}}{2}t\right) + B \sin\left(\frac{\sqrt{3}}{2}t\right) \right].$$

Before we can apply the second condition, we must determine the derivative of x . Applying the Product Rule gives

$$\begin{aligned} \frac{dx}{dt} &= -\frac{1}{2}e^{-\frac{1}{2}t} \left[h \cos\left(\frac{\sqrt{3}}{2}t\right) + B \sin\left(\frac{\sqrt{3}}{2}t\right) \right] \\ &+ e^{-\frac{1}{2}t} \left[-h\frac{\sqrt{3}}{2} \sin\left(\frac{\sqrt{3}}{2}t\right) + B\frac{\sqrt{3}}{2} \cos\left(\frac{\sqrt{3}}{2}t\right) \right]. \end{aligned}$$

Now apply the second condition, $\frac{dx}{dt}(0) = 0$:

$$\begin{aligned} 0 &= -\frac{1}{2}e^{-\frac{1}{2} \times 0} \left[h \cos\left(\frac{\sqrt{3}}{2} \cdot 0\right) + B \sin\left(\frac{\sqrt{3}}{2} \cdot 0\right) \right] \\ &+ e^{-\frac{1}{2} \times 0} \left[-h\frac{\sqrt{3}}{2} \sin\left(\frac{\sqrt{3}}{2} \cdot 0\right) + B\frac{\sqrt{3}}{2} \cos\left(\frac{\sqrt{3}}{2} \cdot 0\right) \right] \end{aligned}$$

$$0 = -\frac{1}{2}h + \frac{\sqrt{3}}{2}B$$

$$B = \frac{h}{\sqrt{3}}.$$

Substitute this value to give the final form for the displacement, x :

$$\underline{\underline{x = e^{-\frac{1}{2}t} \left[h \cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{h}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2}t\right) \right]}}$$

or

$$\underline{\underline{x = e^{-\frac{1}{2}t} \left[h \cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{\sqrt{3}h}{3} \sin\left(\frac{\sqrt{3}}{2}t\right) \right]}}$$

E14. Solve $\frac{d^2y}{dx^2} + 4y = 0$ subject to, $y(0) = 1$ and $\frac{dy}{dx}(0) = 4$.

Auxiliary equation: $\lambda^2 + 4 = 0$

$$\lambda^2 = -4$$

$$\lambda = 0 \pm 2j \quad (\text{complex roots})$$

$$\alpha = 0 \quad \beta = 2$$

General solution: $y = e^{\alpha x} [A \cos(\beta x) + B \sin(\beta x)]$

$$y = e^{0x} [A \cos(2x) + B \sin(2x)]$$

$$\underline{\underline{y = A \cos(2x) + B \sin(2x)}}$$

Apply the first condition: $y(0) = 1$: $1 = A \cos(2 \times 0) + B \sin(2 \times 0)$

$$A = 1 .$$

Before we can apply the second condition we must differentiate the general solution:

$$\frac{dy}{dx} = -2A \sin(2x) + 2B \cos(2x) .$$

Apply the second condition: $\frac{dy}{dx}(0) = 4$:

$$4 = -2A \sin(2 \times 0) + 2B \cos(2 \times 0)$$

$$4 = 0 + 2B$$

$$B = 2 .$$

Substituting for A and B into the general solution gives the particular solution:

$$\underline{y = \cos(2x) + 2 \sin(2x)}.$$

6.2. Non-Homogeneous Equations

In this section we look at how to solve second order, linear, non-homogeneous ODEs with constant coefficients, i.e. equations of the form

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = f(x) \quad [12]$$

where a , b and c are constants

The general solution of a non-homogeneous ODE of the form [12] is found as follows:

STEP I Determine the general solution of the corresponding **homogeneous** ODE (i.e. initially set the RHS, $f(x)$, to zero). In this context, we call the solution the **complementary function** (CF) and denote it $y_c(x)$.

STEP II Determine a **particular integral** (PS), which is any solution of the full non-homogeneous ODE, by the **method of undetermined coefficients** (see below) and denote it $y_p(x)$.

STEP III The **general solution** of the full, non-homogeneous equation is then given by

$$y(x) = y_c(x) + y_p(x).$$

6.2.1. The Method of Undetermined Coefficients

The method enables the derivation of a particular integral, $y_p(x)$, based on the form of the non-homogeneous term, $f(x)$, in the ODE. While the method only works for a limited set of $f(x)$ terms (polynomials, exponentials, sines and cosines and combinations of these functions) it is relatively straightforward to use and only involves algebra and basic differentiation. The method is best illustrated by an example. Consider the ODE:

$$\frac{d^2 y}{dx^2} - \frac{dy}{dx} - 2 y = x^2 + 4 . \quad [13]$$

Step I: Calculate the complementary function:

$$\text{AE:} \quad \lambda^2 - \lambda - 2 = 0$$

$$(\lambda - 2)(\lambda + 1) = 0$$

$$\lambda = 2 \quad \lambda = -1 .$$

Two real distinct roots, so the complementary function is,

$$\underline{\underline{y_c(x) = A e^{2x} + B e^{-x} .}}$$

Step II: The aim is to determine a particular integral of the full, non-homogeneous equation. Its format depends on the type of function, $f(x)$, on the RHS of the ODE. Looking at the RHS above we see (in this case) a quadratic. It is therefore reasonable to assume (make an ‘educated guess’) that a particular integral will be of a similar form, namely a quadratic. We therefore write down the **most general form** that a quadratic can take,

$$y_p = a x^2 + b x + c , \quad [14]$$

where a , b and c are the, as yet, **undetermined coefficients**. Our task is now to calculate the values of a , b and c that make [14] a solution of ODE [13]. To do this substitute the general form [14], and its first and second derivatives, into the LHS of ODE [13] and force this new LHS to equal the given RHS.

We have

$$\frac{dy_p}{dx} = 2ax + b \quad [15]$$

$$\frac{d^2y_p}{dx^2} = 2a \quad [16]$$

Substituting [14], [15] and [16] into the LHS of [13] we get

$$[2a] - [2ax + b] - 2[ax^2 + bx + c] = x^2 + 4 \quad .$$

Collecting together the powers of x on the LHS gives

$$(2a - b - 2c) - (2a + 2b)x - 2ax^2 = x^2 + 4 \quad . \quad [17]$$

If [14] is to be a solution of [13] then we must determine a , b and c so that the LHS of [17] equals the RHS. Equating the coefficients of the powers of x on both sides yields three equations in a , b and c :

$$\text{coefficients of } x^2: \quad -2a = 1$$

$$\text{coefficients of } x: \quad -(2a + 2b) = 0$$

$$\text{constant term:} \quad 2a - b - 2c = 4 \quad .$$

Solving these equations gives

$$a = -\frac{1}{2}, \quad b = \frac{1}{2}, \quad c = -\frac{11}{4} \quad .$$

A particular integral is therefore

$$\underline{\underline{y_p(x) = -\frac{1}{2}x^2 + \frac{1}{2}x - \frac{11}{4} \quad .}}$$

Step III: We can now write down the general solution of ODE [13] by adding together the complementary function, $y_c(x)$, and the particular integral, $y_p(x)$ i.e.

$$y(x) = y_c(x) + y_p(x)$$

so that

$$\underline{\underline{y(x) = A e^{2x} + B e^{-x} - \frac{1}{2} x^2 + \frac{1}{2} x - \frac{11}{4} .}}$$

For other types of RHS functions, $f(x)$, we look for particular integrals which are similar in form and the table on the next page gives the trial function ('educated guess') for $y_p(x)$ corresponding to various RHS functions. The method can run into difficulty however if the non-homogeneous term, $f(x)$, forms part of the complementary function. This situation is addressed in Example 16.

Examples

See Examples (15) – (16) after the table of y_p starting forms.

Table of Trial Functions for Particular Integrals of Non-homogeneous ODEs

| RHS function $f(x)$ | Trial function for $y_p(x)$ |
|---|---|
| Any constant term (e.g. 4 , -3) | a |
| Any linear term (e.g. $2x$, $5x - 8$) | $ax + b$ |
| Any quadratic term (e.g. $3x^2 + 2x - 6$) | $ax^2 + bx + c$ |
| RHS contains an exponential of the form e^{kx} (where k is not a root of the AE) | $a e^{kx}$ |
| RHS contains an exponential of the form e^{kx} (where k is one of two distinct roots of the AE) | $ax e^{kx}$ |
| RHS contains an exponential of the form e^{kx} (where k is a repeated root of the AE) | $ax^2 e^{kx}$ |
| RHS contains $\cos(kx)$ or $\sin(kx)$ or both | $a \cos(kx) + b \sin(kx)$ or $x[a \cos(kx) + b \sin(kx)]$ * |

* This only occurs with ODEs of the form

$$\frac{d^2 y}{dx^2} + k^2 y = \text{RHS containing } \cos(kx) \text{ or } \sin(kx) \text{ or both.}$$

Examples

E15(a). Determine the general solution of the ODE, $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 35y = 12e^{3x}$.

Step I: Calculate the complementary function:

$$\text{AE: } \lambda^2 + 2\lambda - 35 = 0$$

$$(\lambda + 7)(\lambda - 5) = 0$$

$$\lambda = -7 \quad \lambda = 5 .$$

Two real distinct roots, so the complementary function is:

$$\underline{\underline{y_c(x) = A e^{-7x} + B e^{5x} .}}$$

Step II: Determine a particular integral:

Firstly, we note that when $f(x)$ is an exponential function of the form $a e^{kx}$ the nature of the particular integral is determined by whether or not the coefficient (k), appearing in the exponent, is a root of the auxiliary equation.

In this case $f(x) = 12e^{3x}$ so that $k = 3$ and the roots of the AE were found at Step I to be -7 and 5 . Hence, k is not equal to either root and so the table gives that we try

$$y_p = a e^{3x} .$$

Differentiate:

$$y'_p = 3a e^{3x}$$

$$y''_p = 9a e^{3x}$$

Note: In Example 16(a) we look at how to determine the particular integral when the coefficient (k), in the exponent, equals **one** of the roots of the auxiliary equation.

Substitute y_p , y_p' and y_p'' into the ODE and solve for the coefficient, a :

$$9a e^{3x} + 2(3a e^{3x}) - 35(a e^{3x}) = 12e^{3x}$$

$$9a e^{3x} + 6a e^{3x} - 35a e^{3x} = 12e^{3x}$$

$$-20a e^{3x} = 12e^{3x} \quad \Rightarrow \quad a = -\frac{12}{20} = -\frac{3}{5}$$

A particular integral is therefore

$$\underline{\underline{y_p = -\frac{3}{5} e^{3x}}}$$

Step III: Write down the GS of the full, non-homogeneous ODE:

$$\underline{\underline{y = A e^{-7x} + B e^{5x} - \frac{3}{5} e^{3x}}}$$

E15(b). Determine the general solution of the ODE, $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = x + 4$.

Step I: Calculate the complementary function:

$$\text{AE:} \quad \lambda^2 + 5\lambda + 6 = 0$$

$$(\lambda + 2)(\lambda + 3) = 0$$

$$\lambda = -2, \lambda = -3$$

$$\text{CF:} \quad \underline{\underline{y_c = A e^{-2x} + B e^{-3x}}}$$

Step II: Determine a particular integral:

Since the RHS of the ODE is a linear function we select (see the above table)

$$y_p = ax + b.$$

Differentiate

$$y'_p = a$$

$$y''_p = 0.$$

Substitute y_p , y'_p and y''_p into the ODE:

$$0 + 5a + 6(ax + b) = x + 4.$$

Gather like terms:

$$6ax + (5a + 6b) = x + 4.$$

Equate coefficients:

$$x \text{ terms:} \quad 6a = 1$$

$$\text{constant terms:} \quad 5a + 6b = 4.$$

Solve for a and b and substitute into y_p :

$$a = \frac{1}{6}, \quad b = \frac{19}{36}.$$

$$\underline{\underline{y_p = \frac{1}{6}x + \frac{19}{36}}}.$$

Step III: Write down the GS of the full, non-homogeneous ODE:

$$\underline{\underline{y = A e^{-2x} + B e^{-3x} + \frac{1}{6}x + \frac{19}{36}}}.$$

E15(c). Determine the general solution of the ODE, $\frac{d^2y}{dx^2} + y = \sin(2x)$.

Step I: Calculate the complementary function:

AE: $\lambda^2 + 1 = 0$

$$\lambda^2 = -1$$

$$\lambda = \pm j = 0 \pm j. \quad (\alpha = 0, \beta = 1)$$

CF: $\underline{\underline{y_c = A \cos x + B \sin x}}$.

Step II: Determine a particular integral:

From the table we select

$$y_p = a \cos(2x) + b \sin(2x).$$

Note: If the RHS of the ODE contains either sin or cos terms we must have that **both** sin **and** cos appear in y_p , at the same frequency, which in this case is 2.

Differentiate:

$$y_p' = -2a \sin(2x) + 2b \cos(2x)$$

$$y_p'' = -4a \cos(2x) - 4b \sin(2x)$$

Substitute y_p , y_p' and y_p'' into the ODE:

$$[-4a \cos(2x) - 4b \sin(2x)] + [a \cos(2x) + b \sin(2x)] = \sin(2x).$$

Collect like terms:

$$-3a \cos(2x) - 3b \sin(2x) = \sin(2x).$$

Equate coefficients and solve for a and b :

$$\text{cosine terms:} \quad -3a = 0 \quad \Rightarrow \quad a = 0$$

$$\text{sine terms:} \quad -3b = 1 \quad \Rightarrow \quad b = -\frac{1}{3}.$$

A particular integral is therefore

$$\underline{\underline{y_p = -\frac{1}{3}\sin(2x)}}.$$

Step III: Write down the GS of the full, non-homogeneous ODE:

$$\underline{\underline{y = A \cos x + B \sin x - \frac{1}{3}\sin(2x)}}.$$

Now watch the following:

<https://www.youtube.com/watch?v=st8sU-0cgGw> (**Constant function on RHS of ODE**)

<https://www.youtube.com/watch?v=iTxiBnbDwRE> (**Linear function on RHS of ODE**)

<https://www.youtube.com/watch?v=coC-msZupHE> (**Quadratic function on RHS of ODE**)

<https://www.youtube.com/watch?v=oLSaWZ80aKc&t=64s> (**Exp function on RHS of ODE**)

<https://www.youtube.com/watch?v=sZaYXEAUPWY> (**Trig function on RHS of ODE**)

If the non-homogeneous term $f(x)$ appears in the complementary function the method described above will fail. The following examples demonstrate how to overcome this problem.

E16(a). Determine the general solution of the ODE, $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 10y = 6e^{5x}$.

Step I: Calculate the complementary function:

$$\text{AE: } \lambda^2 - 7\lambda + 10 = 0$$

$$(\lambda - 2)(\lambda - 5) = 0$$

$$\lambda = 2, \lambda = 5$$

$$\text{CF: } \underline{\underline{y_c = A e^{2x} + B e^{5x}}}$$

Step II: Determine a particular integral: (c.f. Example 15(a))

The RHS of the ODE contains e^{5x} which has a 5 in the exponent and 5 is one of the two distinct roots of the auxiliary equation. Hence, we choose

$$y_p = a x e^{5x} \quad (\text{ see the 'Table of Trial Functions' })$$

(Alternatively, note that the RHS of the ODE contains e^{5x} which appears in the CF)

Differentiate:

$$y'_p = a [1 \cdot e^{5x} + x(5e^{5x})] = a [e^{5x} + 5xe^{5x}]$$

$$y''_p = a [5e^{5x} + (5e^{5x} + 25xe^{5x})] = a [10e^{5x} + 25xe^{5x}].$$

Substitute y_p , y_p' and y_p'' into the ODE and solve for the coefficient, a :

$$a[10e^{5x} + 25xe^{5x}] - 7a[e^{5x} + 5xe^{5x}] + 10ax e^{5x} = 6e^{5x}$$

$$10ae^{5x} + 25axe^{5x} - 7ae^{5x} - 35axe^{5x} + 10axe^{5x} = 6e^{5x}$$

$$3ae^{5x} = 6e^{5x}$$

$$a = 2$$

A particular integral is therefore

$$\underline{\underline{y_p = 2xe^{5x}}}$$

Step III: Write down the GS of the full, non-homogeneous ODE:

$$\underline{\underline{y = Ae^{2x} + Be^{5x} + 2xe^{5x}}}$$

Now watch the following: https://www.youtube.com/watch?v=PPRLsNQCw_w

E16(b). Determine the general solution of the ODE, $\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 25y = 6e^{5x}$.

Step I: Calculate the complementary function:

$$\text{AE: } \lambda^2 - 10\lambda + 25 = 0$$

$$(\lambda - 5)(\lambda - 5) = 0$$

$$\lambda = 5 \text{ (twice)}$$

$$\text{CF: } \underline{\underline{y_c = (A + Bx)e^{5x}}}$$

Step II: Determine a particular integral: (c.f. Examples 15(a) and 16(a))

The RHS of the ODE contains e^{5x} which has a 5 in the exponent and equals the repeated root of the auxiliary equation. Hence, we choose

$$y_p = a x^2 e^{5x} \quad (\text{ see the 'Table of Trial Functions' })$$

(Alternatively, the RHS of the ODE contains e^{5x} which appears twice in the CF)

Exercise: Try to complete this example to show that the particular integral is

$$\underline{\underline{y_p = 3 x^2 e^{5x}}} \quad \text{and that the general solution is}$$

$$\underline{\underline{y = (A + Bx) e^{5x} + 3 x^2 e^{5x} .}}$$

E16(c). Determine the general solution of the ODE, $\frac{d^2y}{dx^2} + 9y = \sin(3x)$.

Step I: Calculate the complementary function:

AE: $\lambda^2 + 9 = 0$

$$\lambda^2 = -9$$

$$\lambda = \pm 3j = 0 \pm 3j \quad (\alpha = 0, \beta = 3)$$

CF: $\underline{\underline{y_c = A \cos(3x) + B \sin(3x)}}$

Step II: Determine a particular integral:

The table gives that we should try

$$y_p = x [a \cos(3x) + b \sin(3x)] . \quad (\text{ see note at end })$$

Exercise: Try to complete this example to show that that the particular integral is

$$\underline{\underline{y_p = -\frac{1}{6} x \cos(3x)}} \quad \text{and that the general solution is}$$

$$\underline{\underline{y = A \cos(3x) + B \sin(3x) - \frac{1}{6} x \cos(3x).}}$$

Note: In this case as the frequency (3) of the trig function on the RHS of the ODE matches exactly the frequency (3) of the trig functions in the complementary function. Hence, we must try a particular integral of the form selected above.

6.2.2. Non-Homogeneous ODE with Initial Conditions

Note that if we are given an ODE with initial conditions we can calculate the values of A and B and completely solve the ODE. This type of problem, when all the constraints are specified at the same value of the independent variable (x), is known as an **initial value problem (IVP)**. In this case we must derive the full general solution before applying the initial conditions to find A and B .

E17. Solve the IVP, $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 35y = 12e^{3x}$, $y(0) = 2$, $y'(0) = 0$.

We obtained the general solution in Example 15(a) as,

$$y = A e^{-7x} + B e^{5x} - \frac{3}{5} e^{3x}.$$

You should check that applying the specified conditions gives $A = \frac{14}{15}$ and $B = \frac{5}{3}$ so that the full solution is,

$$\underline{\underline{y = \frac{14}{15} e^{-7x} + \frac{5}{3} e^{5x} - \frac{3}{5} e^{3x} .}}$$

Now watch the following: <https://www.youtube.com/watch?v=QvOk8CgUuXk&t=54s>

Summary

This unit has introduced the concept of an ordinary differential equation (ODE). You should now be able to:

- classify differential equations according to their order and linearity.
- solve differential equations by direct integration.
- solve first order separable differential equations.
- solve first order linear differential equations using the integrating factor method.
- solve second-order linear constant coefficient differential equations with standard forcing functions using the method of undetermined coefficients.

In the next unit we extend ideas from integration of a function of a single variable to integrating functions of two variables. We look at the interpretation of a double integral as a volume, changing the order of integration and converting from Cartesian to polar coordinates.

Tutorial Exercises

First Order ODEs – Separable

Q1. Determine the general solution of each of the following ODEs:

(i). $\frac{dy}{dx} - 2y = 0$

(ii). $\frac{dy}{dx} + 4y^2 = 0$

(iii). $\frac{dy}{dx} - 2y = 4$

(iv). $x \frac{dy}{dx} + y = 0$

(v). $\frac{dy}{dx} + e^x y^2 = 0$

(vi). $\frac{dy}{dx} = 2x^{-1} \sqrt{y-1}$.

Q2. Determine the particular integral of each of the following ODEs:

(i). $(x+1) \frac{dy}{dx} = 2y$, $y(0) = 1$

(ii). $\frac{dy}{dx} + e^{2x} y^2 = 0$, $y(0) = -1$

(iii). $2t \frac{dx}{dt} = 3x$, $x(1) = 4$.

First Order ODEs – Integrating Factors

Q3. Determine the general solution of each of the following ODEs:

(i). $\frac{dy}{dx} - y = 3$

(ii). $\frac{dy}{dx} + 2y = 6e^x$

(iii). $\frac{dy}{dx} + 2xy = x e^{-x^2}$

(iv). $\frac{dy}{dx} + 2xy = 2x$

(v). $x \frac{dy}{dx} + y = x + x^3$

(vi). $x \frac{dy}{dx} = y + (x+1)^2$.

Q4. Determine the particular integral of each of the following ODEs:

(i). $\frac{dy}{dx} + y = e^{-x}$, $y(0) = 1$

(ii). $\frac{dy}{dx} - \frac{2}{x}y = x^2 \cos(x)$, $y(\pi) = 3\pi^2$.

Second Order ODEs – Linear, Constant Coefficients (Homogeneous)

Q5. Determine the general solution of each of the following ODEs:

(i). $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$ (ii). $2\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 2y = 0$

(iii). $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$ (iv). $4\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 9y = 0$

(v). $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = 0$ (vi). $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$

(vii). $\frac{d^2y}{dx^2} - 9y = 0$ (viii). $\frac{d^2y}{dx^2} + 9y = 0$.

Q6. Determine the particular integral of each of the following ODEs:

(i). $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 8y = 0$, $y(0) = 0$ $y'(0) = 6$

(ii). $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 13y = 0$, $y(0) = 2$ $y'(0) = 0$.

Second Order ODEs – Linear, Constant Coefficients (Non-Homogeneous)

Q7. Determine the general solution of each of the following ODEs:

(i). $\frac{d^2y}{dx^2} + 10\frac{dy}{dx} + 16y = 80$

(ii). $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 3x + 3$

(iii). $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 4y = -8x - 6$

(iv). $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 2e^{2x}$.

Q8. Determine the particular integral of each of the following ODEs:

(i). $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 6e^{2x}$, $y(0) = 0$ $y'(0) = 10$

(ii). $\frac{d^2y}{dx^2} - y = 2$, $y(0) = 0$ $y'(0) = 0$.

Tutorial Answers

A1. (i). $y = A e^{2x}$

(ii). $y = \frac{1}{(4x - C)}$

(iii). $y = A e^{2x} - 2$

(iv). $y = \frac{A}{x}$

(v). $y = \frac{1}{e^x - C}$

(vi). $y = [\ln|x| + C]^2 + 1$.

A2. (i). $y = (x + 1)^2$

(ii). $y = \frac{2}{e^{2x} - 3}$

(iii). $x^2 = 16 t^3.$

A3. (i). $r(x) = e^{-x}$

$y = A e^x - 3$

(ii). $r(x) = e^{2x}$

$y = 2 e^x + A e^{-2x}$

(iii). $r(x) = e^{x^2}$

$y = (\frac{1}{2}x^2 + C) e^{-x^2}$

(iv). $r(x) = e^{-x^2}$

$y = 1 + A e^{-x^2}$

(v). $r(x) = x$

$y = \frac{1}{2}x + \frac{1}{4}x^3 + \frac{A}{x}$

(vi). $r(x) = \frac{1}{x}$

$y = x^2 + 2x \ln|x| + Ax - 1.$

A4. (i). $r(x) = e^x$

$y = (x+1)e^{-x}$

(ii). $r(x) = x^{-2}$

$y = x^2 \sin(x) + 3x^2.$

A5. (i). $y = A e^{3x} + B e^{-2x}$

(ii). $y = A e^{0.5x} + B e^{2x}$

(iii). $y = (A + Bx) e^{3x}$

(iv). $y = (A + Bx) e^{1.5x}$

(v). $y = e^x [A \cos(\sqrt{3} x) + B \sin(\sqrt{3} x)]$

$$\text{(vi). } y = e^{-x} [A \cos(x) + B \sin(x)] \quad \text{(vii). } y = A e^{3x} + B e^{-3x}$$

$$\text{(viii). } y = A \cos(3x) + B \sin(3x).$$

$$\text{A6. (i). } y = e^{4x} - e^{-2x} \quad \text{(ii). } y = e^{-3x} [2 \cos(2x) + 3 \sin(2x)].$$

$$\text{A7. (i). } y = A e^{-8x} + B e^{-2x} + 5$$

$$\text{(ii). } y = A e^{2x} + B e^{3x} + \frac{1}{2}x + \frac{11}{12}$$

$$\text{(iii). } y = A e^{-4x} + B e^x + 2x + 3$$

$$\text{(iv). } y = A e^{-x} + B e^{-2x} + \frac{1}{6}e^{2x}.$$

$$\text{A8. (i). } y = 4e^{3x} - 2e^{-x} - 2e^{2x}$$

$$\text{(ii). } y = e^x + e^{-x} - 2.$$