Separable ODEs: Question (1) (i) : COMPLETE SOLUTION

1. $$\frac{dy}{dx} - 2y = 0$$

2. $$\frac{dy}{dx} = 2y$$

3. $$\frac{dy}{y} = 2\,dx$$

4. $$\int \frac{1}{y}\,dx = \int 2\,dx$$

5. $$\ln|\,y\,| = 2x + C$$

6. $$y = e^{2x+C}$$

7. $$y = e^{2x}\,e^{C}$$

8. $$\underline{\underline{y = A\,e^{2x}}}$$

Separable ODEs: Question (1) (ii) : COMPLETE SOLUTION

1. $$\frac{dy}{dx} + 4y^2 = 0$$

2. $$\frac{dy}{dx} = -4y^2$$

3. $$\frac{dy}{y^2} = -4\ dx$$

4. $$\int \frac{1}{y^2}\ dy = \int (-4)\ dx$$

5. $$\int y^{-2}\ dy = -4x + C$$

6. $$-y^{-1} = -4x + C$$

7. $$\frac{1}{y} = 4x - C$$

8. $$y = \frac{1}{4x - C}$$

Separable ODEs: Question (1) (iii) : COMPLETE SOLUTION

1. $$\frac{dy}{dx} - 2y = 4$$

2. $$\frac{dy}{dx} = 2y + 4$$

3. $$\frac{dy}{2y+4} = dx$$

4. $$\int \frac{1}{2y+4}\,dy = \int 1\,dx$$

5. $$\tfrac{1}{2}\ln\left|\,2y+4\,\right| = x + C$$

6. $$\ln\left|\,2y+4\,\right| = 2x + 2C$$

7. $$2y + 4 = e^{2x+2C}$$

8. $$2y = e^{2x}e^{2C} - 4$$

9. $$y = \tfrac{1}{2}e^{2C}e^{2x} - 2$$

10. $$\underline{\underline{y = Ae^{2x} - 2}}$$

Separable ODEs: Question (1) (iv) : COMPLETE SOLUTION

1. $$x \frac{dy}{dx} + y = 0$$

2. $$x \frac{dy}{dx} = -y$$

3. $$\frac{dy}{y} = -\frac{dx}{x}$$

4. $$\int \frac{1}{y}\, dy = -\int \frac{1}{x}\, dx$$

5. $$\ln|y| = -\ln|x| + C$$

6. $$\ln|y| + \ln|x| = C$$

7. $$\ln|yx| = C$$

8. $$yx = e^{C}$$

9. $$y = \frac{A}{x}$$

Separable ODEs: Question (1) (v) : COMPLETE SOLUTION

1. $$(x+2)\frac{dy}{dx} - xy = 0$$

2. $$(x+2)\frac{dy}{dx} = xy$$

3. $$\frac{dy}{y} = \frac{x}{x+2}\,dx$$

4. $$\int \frac{1}{y}\,dy = \int \left[\frac{x}{x+2}\right] dx$$

5. $$\ln|y| = \int \left[1 - \frac{2}{x+2}\right] dx$$

6. $$\ln|y| = x - 2\ln|x+2| + C$$

7. $$\ln|y| + 2\ln|x+2| = x + C$$

8. $$\ln|y| + \ln|(x+2)^2| = x + C$$

9. $$\ln|y(x+2)^2| = x + C$$

10. $$y(x+2)^2 = e^{x+C}$$

11. $$y(x+2)^2 = e^x e^C$$

12. $$y = \frac{A e^x}{(x+2)^2}$$

Separable ODEs: Question (2) (i) : COMPLETE SOLUTION

1. $$(x+1)\frac{dy}{dx} = 2y$$

2. $$\frac{dy}{y} = \frac{2}{(x+1)}\,dx$$

3. $$\int \frac{1}{y}\,dy = \int \frac{2}{(x+1)}\,dx$$

4. $$\ln|y| = 2\ln|x+1| + C$$

Apply condition $y(0) = 1$

5. $$\ln|1| = 2\ln|0+1| + C$$

6. $$C = 0$$

Input value

7. $$\ln|y| = 2\ln|x+1| + 0$$

8. $$\ln|y| - \ln|(x+1)^2| = 0$$

9. $$\ln\left|\frac{y}{(x+1)^2}\right| = 0$$

10. $$\frac{y}{(x+1)^2} = 1$$

11. $$y = (x+1)^2$$

Separable ODEs: Question (2) (ii) : COMPLETE SOLUTION

1. $$\frac{dy}{dx} = y\tan(2x)$$

2. $$\frac{dy}{y} = \tan(2x)\, dx$$

3. $$\int \frac{1}{y}\, dy = \int \tan(2x)\, dx$$

4. $$\ln|y| = -\tfrac{1}{2}\ln|\cos(2x)| + C$$

Apply condition $y(0) = 2$

5. $$\ln|2| = -\tfrac{1}{2}\ln|\cos(2\times 0)| + C$$

6. $$C = \ln|2|$$

Input value

7. $$\ln|y| = -\tfrac{1}{2}\ln|\cos(2x)| + \ln|2|$$

8. $$\ln\left|\frac{y[\cos(2x)]^{1/2}}{2}\right| = 0$$

9. $$\frac{y[\cos(2x)]^{1/2}}{2} = 1$$

10. $$y = \frac{2}{\sqrt{\cos(2x)}}$$

Integrating Factor ODEs: Question (3) (i)

$$\frac{dy}{dx} - y = 3$$

Determination of Integrating Factor:

$$p(x) = -1$$

$$r(x) = e^{\int p(x)\,dx}$$
$$= e^{\int (-1)\,dx}$$
$$= e^{-x}$$

Final Steps:

$$e^{-x}\frac{dy}{dx} - e^{-x}y = 3e^{-x}$$

$$\frac{d}{dx}\left[e^{-x}y\right] = 3e^{-x}$$

$$e^{-x}y = \int 3e^{-x}\,dx$$
$$= -3e^{-x} + C$$

$$y = -3 + Ce^{+x}$$

$$\underline{\underline{y = Ce^{+x} - 3}}$$

Integrating Factor ODEs: Question (3) (ii)

$$\frac{dy}{dx} + 2y = 6e^{x}$$

Determination of Integrating Factor:

$$p(x) = +2$$

$$\begin{aligned} r(x) &= e^{\int p(x)\,dx} \\ &= e^{\int (+2)\,dx} \\ &= e^{2x} \end{aligned}$$

Final Steps:

$$e^{2x}\frac{dy}{dx} + 2e^{2x}y = 6e^{3x}$$

$$\frac{d}{dx}\left[e^{2x}y\right] = 6e^{3x}$$

$$\begin{aligned} e^{2x}y &= \int 6e^{3x}\,dx \\ &= 2e^{3x} + C \end{aligned}$$

$$\underline{\underline{y = 2e^{x} + Ce^{-2x}}}$$

Integrating Factor ODEs: Question (3) (iii)

$$\frac{dy}{dx} + 2xy = x e^{-x^2}$$

Determination of Integrating Factor:

$$p(x) = 2x$$

$$r(x) = e^{\int p(x)\,dx}$$
$$= e^{\int (2x)\,dx}$$
$$= e^{x^2}$$

Final Steps:

$$e^{x^2}\frac{dy}{dx} + 2x e^{x^2} y = x$$

$$\frac{d}{dx}\left[e^{x^2} y \right] = x$$

$$e^{x^2} y = \int x\,dx$$
$$= \tfrac{1}{2}x^2 + C$$

$$y = \left(\tfrac{1}{2}x^2 + C \right) e^{-x^2}$$

Integrating Factor ODEs: Question (3) (iv)

$$\frac{dy}{dx} + 2xy = 2x$$

Determination of Integrating Factor:

$$p(x) = 2x$$

$$r(x) = e^{\int p(x)\,dx}$$
$$= e^{\int (2x)\,dx}$$
$$= e^{x^2}$$

Final Steps:

$$e^{x^2}\frac{dy}{dx} + 2x\,e^{x^2}\,y = 2x\,e^{x^2}$$

$$\frac{d}{dx}\left[e^{x^2}\,y\right] = 2x\,e^{x^2}$$

$$e^{x^2}\,y = \int e^{x^2}\,2x\,dx$$

$$u = x^2$$
$$\frac{du}{dx} = 2x$$
$$du = 2x\,dx$$

$$e^{x^2}\,y = \int e^{u}\,du$$
$$= e^{u} + C$$
$$= e^{x^2} + C$$

$$\underline{\underline{y = 1 + C\,e^{-x^2}}}$$

Integrating Factor ODEs: Question (3) (v)

$$x \frac{dy}{dx} + y = x + x^3$$

$$\frac{dy}{dx} + \frac{1}{x} y = 1 + x^2$$

Determination of Integrating Factor:

$$p(x) = \frac{1}{x}$$

$$\begin{aligned} r(x) &= e^{\int p(x)\,dx} \\ &= e^{\int \frac{1}{x}\,dx} \\ &= e^{\ln|x|} \\ &= x \end{aligned}$$

Final Steps:

$$x \frac{dy}{dx} + y = x + x^3$$

$$\frac{d}{dx}[\,xy\,] = x + x^3$$

$$\begin{aligned} xy &= \int \left(x + x^3\right) dx \\ &= \tfrac{1}{2}x^2 + \tfrac{1}{4}x^4 + C \end{aligned}$$

$$y = \tfrac{1}{2}x + \tfrac{1}{4}x^3 + \frac{C}{x}$$

Integrating Factor ODEs: Question (3) (vi)

$$x\frac{dy}{dx} = y + (x+1)^2$$

$$x\frac{dy}{dx} - y = (x+1)^2$$

$$\frac{dy}{dx} - \frac{1}{x}y = \frac{(x+1)^2}{x}$$

Determination of Integrating Factor:

$$p(x) = -\frac{1}{x}$$

$$\begin{aligned} r(x) &= e^{\int p(x)\,dx} \\ &= e^{\int \left(-\frac{1}{x}\right)dx} \\ &= e^{-\ln|x|} \\ &= e^{\ln|x^{-1}|} \\ &= x^{-1} \\ &= \frac{1}{x} \end{aligned}$$

Final Steps:

$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = \frac{(x+1)^2}{x^2}$$

$$\frac{d}{dx}\left[\frac{1}{x}y\right] = \frac{(x+1)^2}{x^2}$$

$$\begin{aligned} \frac{y}{x} &= \int \frac{(x+1)^2}{x^2}\,dx \\ &= \int \frac{x^2+2x+1}{x^2}\,dx \\ &= \int \left(1 + \frac{2}{x} + \frac{1}{x^2}\right)dx \\ &= x + 2\ln|x| - \frac{1}{x} + C \end{aligned}$$

$$y = x^2 + 2x\ln|x| + Cx - 1$$

Integrating Factor ODEs: Question (3) (vii)

$$\frac{dy}{dx} - \frac{2}{x} y = x^2 e^x$$

Determination of Integrating Factor:

$$p(x) = -\frac{2}{x}$$

$$\begin{aligned} r(x) &= e^{\int p(x)\, dx} \\ &= e^{-\int \frac{2}{x}\, dx} \\ &= e^{-2 \ln|x|} \\ &= e^{\ln|x^{-2}|} \\ &= x^{-2} \end{aligned}$$

Final Steps:

$$x^{-2} \frac{dy}{dx} - 2x^{-3} y = e^x$$

$$\frac{d}{dx}\left[x^{-2} y \right] = e^x$$

$$\begin{aligned} x^{-2} y &= \int e^x \, dx \\ &= e^x + C \end{aligned}$$

$$\underline{\underline{y = x^2 \left(e^x + C \right)}}$$

Integrating Factor ODEs: Question (3) (viii)

$$\frac{dy}{dx} + y = x$$

Determination of Integrating Factor:

$$p(x) = 1$$

$$r(x) = e^{\int p(x)\,dx}$$
$$= e^{\int 1\,dx}$$
$$= e^{x}$$

Final Steps:

$$e^{x}\frac{dy}{dx} + e^{x}y = e^{x}x$$

$$\frac{d}{dx}\left[e^{x}y\right] = e^{x}x$$

$$e^{x}y = \int e^{x}x\,dx$$
$$= e^{x}x - \int e^{x}.1\,dx$$
$$= e^{x}x - e^{x} + C$$
$$= (x-1)e^{x} + C$$

$$\underline{\underline{y = (x-1) + Ce^{-x}}}$$

Integrating Factor ODEs: Question (4) (i)

$$\frac{dy}{dx} + y = (x+1)^2$$

Determination of Integrating Factor:

$$p(x) = 1$$

$$r(x) = e^{\int p(x)\,dx}$$
$$= e^{\int 1\,dx}$$
$$= e^{x}$$

Key Final Steps:

$$e^{x} y = \int e^{x} (x+1)^2 \, dx$$

Repeated integration by parts leads to

$$e^{x} y = e^{x} [(x+1)^2 - 2(x+1) + 2] + C$$

Apply condition $y(0) = 0$:

$$e^{0}\, 0 = e^{0} \left[(0+1)^2 - 2(0+1) + 2] + C\right]$$

$$0 = 1 + C$$

$$C = -1$$

Substitute value and solve for y:

$$e^{x} y = e^{x} [(x+1)^2 - 2(x+1) + 2] - 1$$

$$\underline{\underline{y = (x+1)^2 - 2(x+1) + 2 - e^{-x}}}$$

Integrating Factor ODEs: Question (4) (ii)

$$\frac{dy}{dx} - 2y = e^{2x} + e^{-2x}$$

Determination of Integrating Factor:

$$p(x) = -2$$

$$\begin{aligned} r(x) &= e^{\int p(x)\,dx} \\ &= e^{\int (-2)\,dx} \\ &= e^{-2x} \end{aligned}$$

Key Final Steps:

$$e^{-2x} y = \int (1 + e^{-4x})\,dx$$

This leads to

$$y = x e^{2x} - \tfrac{1}{4} e^{-2x} + C e^{2x}$$

Apply condition $y(0) = -2$:

$$-2 = 0 e^{2 \times 0} - \tfrac{1}{4} e^{-2 \times 0} + C e^{2 \times 0}$$

$$-2 = -\tfrac{1}{4} + C$$

$$C = -\tfrac{7}{4}$$

Substitute value:

$$\underline{\underline{y = x e^{2x} - \tfrac{1}{4} e^{-2x} - \tfrac{7}{4} e^{2x}}}$$

Integrating Factor ODEs: Question (4) (iii)

$$\frac{dy}{dx} + y = x$$

Determination of Integrating Factor:

$$p(x) = 1$$

$$r(x) = e^{\int p(x)\,dx}$$
$$= e^{\int 1\,dx}$$
$$= e^{x}$$

Key Final Steps:

$$e^{x} y = \int e^{x} x\,dx$$

Integration by parts leads to

$$y = x - 1 + C e^{-x}$$

Apply condition $y(0) = 0$:

$$0 = 0 - 1 + C e^{-0}$$

$$0 = -1 + C$$

$$C = 1$$

Substitute value:

$$\underline{\underline{y = x - 1 + e^{-x}}}$$

Integrating Factor ODEs: Question (4) (iv)

$$x\,\frac{dy}{dx} \;-\; 3y \;=\; x^6$$

$$\frac{dy}{dx} \;-\; \frac{3}{x}y \;=\; x^5$$

Determination of Integrating Factor:

$$p(x) \;=\; -\frac{3}{x}$$

$$\begin{aligned} r(x) &= e^{\int p(x)\,dx} \\ &= e^{\int \left(\frac{3}{x}\right)dx} \\ &= e^{-3\ln|x|} \\ &= e^{\ln|x^{-3}|} \\ &= x^{-3} \end{aligned}$$

Key Final Steps:

$$x^{-3}y \;=\; \int x^2\;dx$$

This leads to

$$y \;=\; \tfrac{1}{3}x^6 \;+\; C\,x^3$$

Apply condition $y(1) \;=\; 2$:

$$2 \;=\; \tfrac{1}{3}1^6 \;+\; C\,1^3$$

$$2 \;=\; \tfrac{1}{3} \;+\; C$$

$$C \;=\; \tfrac{5}{3}$$

Substitute value:

$$y \;=\; \tfrac{1}{3}\,x^6 \;+\; \tfrac{5}{3}\,x^3$$

2nd Order ODEs – Linear, Constant Coefficients (Homogeneous): (5)(i) – (viii)

These should be straightforward.

You may have the terms the other way round, e.g. in part (i) you may have

$$y = A e^{-2x} + B e^{3x}$$

instead of

$$y = A e^{3x} + B e^{-2x},$$

but that is fine.

If an answer doesn't agree, check the coefficients of the auxiliary equation and its solution. Just remember:

Two real roots $\leftrightarrow$ $y = A e^{\lambda_1 x} + B e^{\lambda_2 x}$

Two complex roots $\leftrightarrow$ $y = e^{\alpha x} [A \cos(\beta x) + B \sin(\beta x)]$

One real root $\leftrightarrow$ $y = (A + B x) e^{\lambda_0 x}$.

2nd Order ODEs – Linear, Constant Coefficients (Homogeneous): (6) (i)

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 8y = 0$$

Auxiliary equation has two real roots.

General solution: $y = A e^{4x} + B e^{-2x}$

Apply condition $y(0) = 0$:

$$0 = A e^{4 \times 0} + B e^{-2 \times 0}$$

$$\underline{\underline{A + B = 0}}$$

Before applying the second condition we must find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = 4A e^{4x} - 2B e^{-2x}$$

Now apply condition $y'(0) = 6$:

$$6 = 4A e^{4 \times 0} - 2B e^{-2 \times 0}$$

$$\underline{\underline{4A - 2B = 6}}$$

Solve the two underlined equations simultaneously to give:

$$A = 1 \quad , \quad B = -1$$

Substitute into general solution to give:

$$\underline{\underline{y = e^{4x} - e^{-2x}}}$$

2nd Order ODEs – Linear, Constant Coefficients (Homogeneous): (6) (ii)

$$\frac{d^2y}{dx^2} \;+\; 6\,\frac{dy}{dx} \;+\; 13\,y \;=\; 0$$

Auxiliary equation has two complex roots.

General solution: $y \;=\; e^{-3x}\,[\;A\cos(2x) \;+\; B\sin(2x)\;]$

Apply condition $y(0) \;=\; 2$:

$$2 \;=\; e^{-3\times 0}\,[\;A\cos(2\times 0) \;+\; B\sin(2\times 0)\;]$$

$$\underline{\underline{A \;=\; 2}}$$

Before applying the second condition we must find $\frac{dy}{dx}$ using the product rule:

$$\frac{dy}{dx} \;=\; -3\,e^{-3x}\,[\;A\cos(2x) \;+\; B\sin(2x)\;] \\ \;+\; e^{-3x}\,[\;-2A\sin(2x) \;+\; 2B\cos(2x)\;]$$

Now apply condition $y'(0) \;=\; 0$:

$$0 \;=\; -3\,e^{-3\times 0}\,[\;A\cos(2\times 0) \;+\; B\sin(2\times 0)\;] \\ \;+\; e^{-3\times 0}\,[\;-2A\sin(2\times 0) \;+\; 2B\cos(2\times 0)\;]$$

$$\underline{\underline{-3A \;+\; 2B \;=\; 0}}$$

Solve the two underlined equations simultaneously to give:

$$A \;=\; 2 \quad , \quad B \;=\; 3$$

Substitute into general solution to give:

$$\underline{\underline{y \;=\; e^{-3x}\,[\;2\cos(2x) \;+\; 3\sin(2x)\;]}}$$

2nd Order ODEs – Linear, Constant Coefficients (Homogeneous): (6) (iii)

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$$

Auxiliary equation has only a single root.

General solution: $y = (A + Bx)\,e^{-2x}$

Apply condition $y(0) = 1$:

$$1 = (A + B\times 0)\,e^{-2\times 0}$$

$$\underline{\underline{A = 1}}$$

Note: Second condition is not a derivative condition, so no need to differentiate; apply condition $y(1) = 0$ to general solution:

$$0 = (A + B\times 1)\,e^{-2\times 1}$$

$$\underline{\underline{A + B = 0}}$$

Solve the two underlined equations simultaneously to give:

$$A = 1 \quad , \quad B = -1$$

Substitute into general solution to give:

$$\underline{\underline{y = (1 - x)\,e^{-2x}}}$$

2nd Order ODEs – Linear, Constant Coefficients (Homogeneous): (6) (iv)

$$\frac{d^2y}{dx^2} + 4y = 0$$

Auxiliary equation has complex roots.

General solution: $y = A\cos(2x) + B\sin(2x)$

Apply condition $y(0) = 1$:

$$1 = A\cos(2\times 0) + B\sin(2\times 0)$$

$$\underline{\underline{A = 1}}$$

Note: Second condition is not a derivative condition, so no need to differentiate; apply condition $y(\frac{\pi}{4}) = 3$ to general solution:

$$3 = A\cos(\tfrac{\pi}{2}) + B\sin(\tfrac{\pi}{2})$$

$$\underline{\underline{B = 3}}$$

Substitute into general solution to give:

$$\underline{\underline{y = \cos(2x) + 3\sin(2x)}}$$

2nd Order ODEs – Linear, Constant Coefficients (Homogeneous): (6) (v)

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$$

Auxiliary equation has two real roots.

General solution: $y = A e^{-x} + B e^{-2x}$

Apply condition $y(0) = 0$:

$$0 = A e^{-0} + B e^{-2\times 0}$$

$$\underline{\underline{A + B = 0}}$$

Note: Second condition is not a derivative condition, so no need to differentiate; apply condition $y(1) = 2$ to general solution:

$$2 = A e^{-1} + B e^{-2}$$

$$\underline{\underline{A e^{-1} + B e^{-2} = 2}}$$

From the first underlined equation,

$$B = -A \ .$$

Substitute into second underlined equation:

$$A e^{-1} - A e^{-2} = 2$$
$$A(e^{-1} - e^{-2}) = 2$$
$$A = \frac{2}{(e^{-1} - e^{-2})} \ ,$$

and so

$$B = -\frac{2}{(e^{-1} - e^{-2})}$$

Substitute into general solution to give:

$$y = \frac{2}{(e^{-1} - e^{-2})} e^{-x} - \frac{2}{(e^{-1} - e^{-2})} e^{-2x}$$

or

$$\underline{\underline{y = \frac{2(e^{-x} - e^{-2x})}{(e^{-1} - e^{-2})}}}$$