Spectral Analysis

In particular – how the sampling process affects the frequency spectrum

The Fourier Theorem

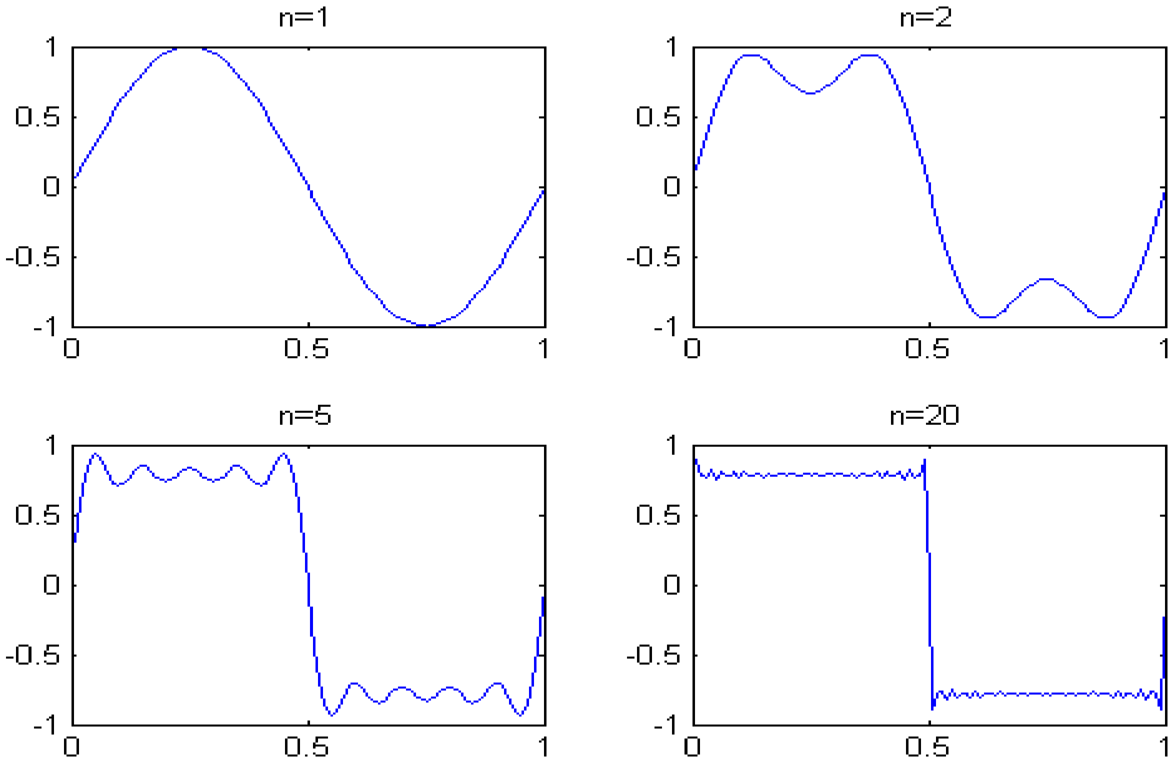
A **mathematical theorem** stating that a **periodic** function f(x) which is reasonably continuous may be expressed as the sum of a series of sine or cosine terms (called the Fourier series), each of which has specific **amplitude** and **phase** coefficients known as Fourier coefficients.

Fourier Series

- Fourier Analysis involves resolving a function into its series of Fourier coefficients.

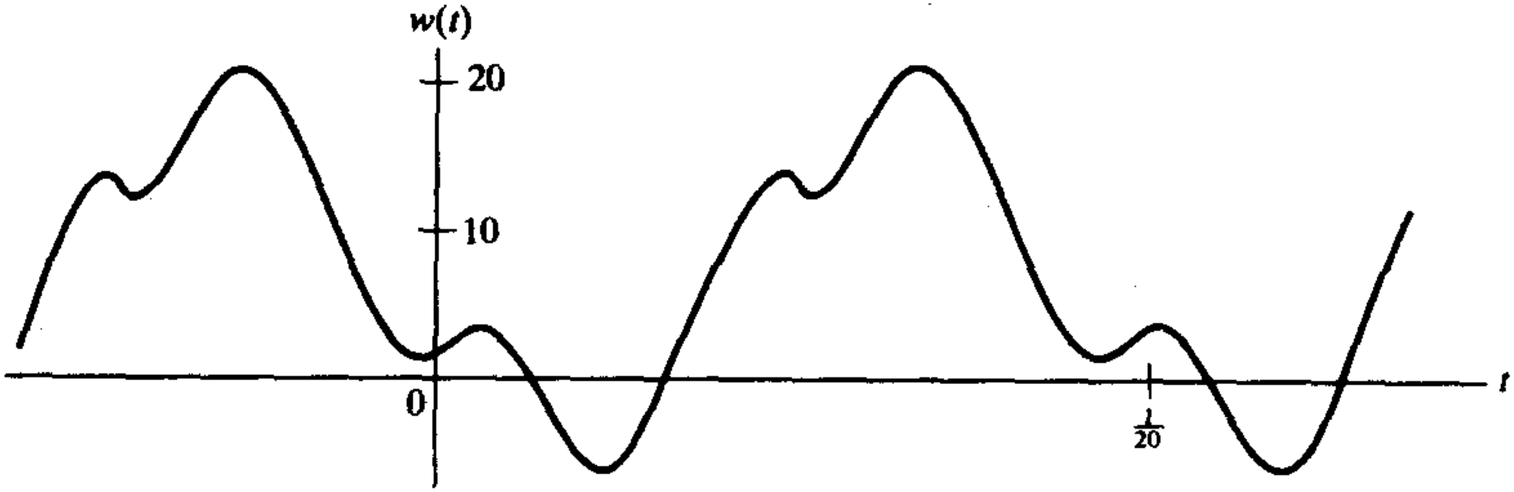
- Fourier Synthesis involves constructing a function from a series of Fourier coefficients.

Fourier Synthesis – a Square Wave



$$x(t) = sin(\omega t) + \frac{1}{3} sin(3\omega t) + \frac{1}{5} sin(5\omega t) + \cdots$$

Fourier Series



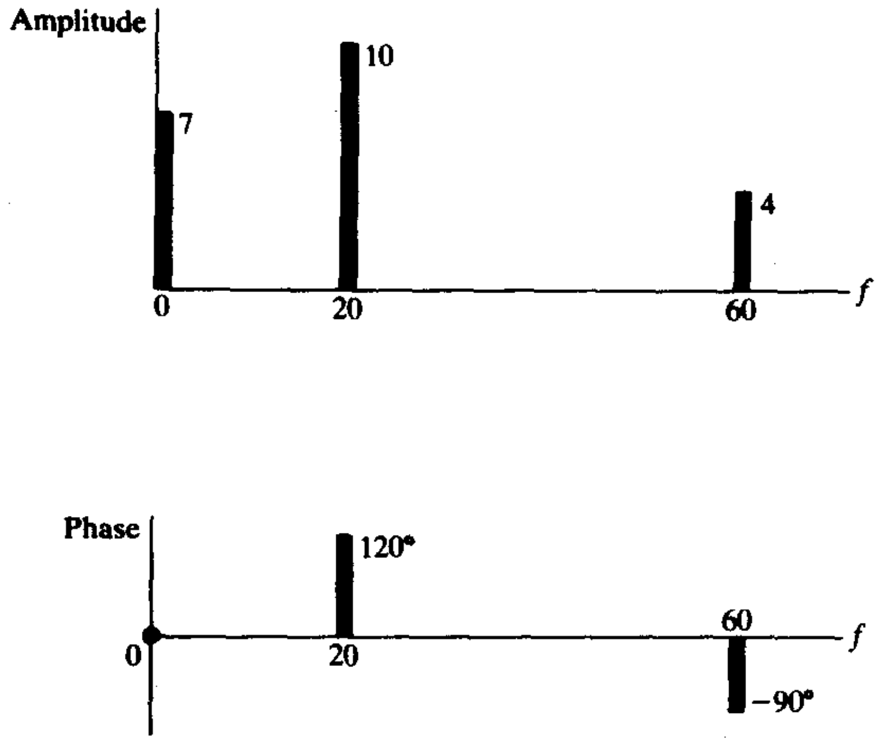
Fourier Series

- Consider the waveform

$$w(t) = 7 - 10\cos(40\pi t - 60^\circ) + 4\sin 120\,\pi t$$

- We can sketch the one sided (positive) frequency spectrum

One Sided (Positive) Spectrum



Two Sided Frequency Spectrum

- Recall the trigonometric identity

$$A\cos(\omega t+\phi) = \tfrac{1}{2}A\, e^{j\phi}\, e^{j\omega t} + \tfrac{1}{2}A\, e^{-j\phi}\, e^{-j\omega t}$$

- So a real cosine wave can be represented as a pair of conjugate exponential functions.
- These can be shown on a phasor diagram.
- Note the negative(!) frequencies

Interpretation of negative frequencies

When $\omega > 0$ the real part of the function leads the imaginary part

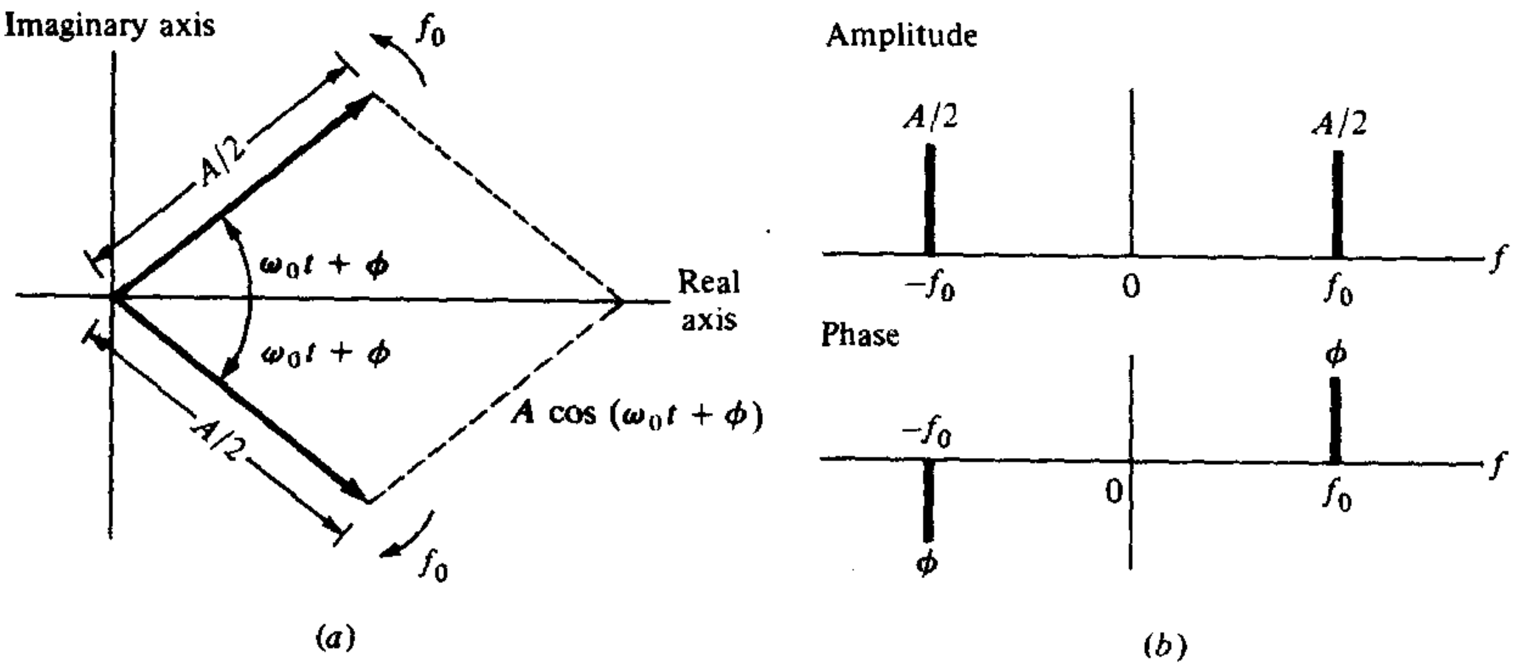
$$e^{j\omega t} = \cos\omega t + j\sin\omega t$$

When $\omega < 0$ the real part of the function lags the imaginary part

$$e^{-j\omega t} = \cos(-\omega t) + j\sin(-\omega t)$$

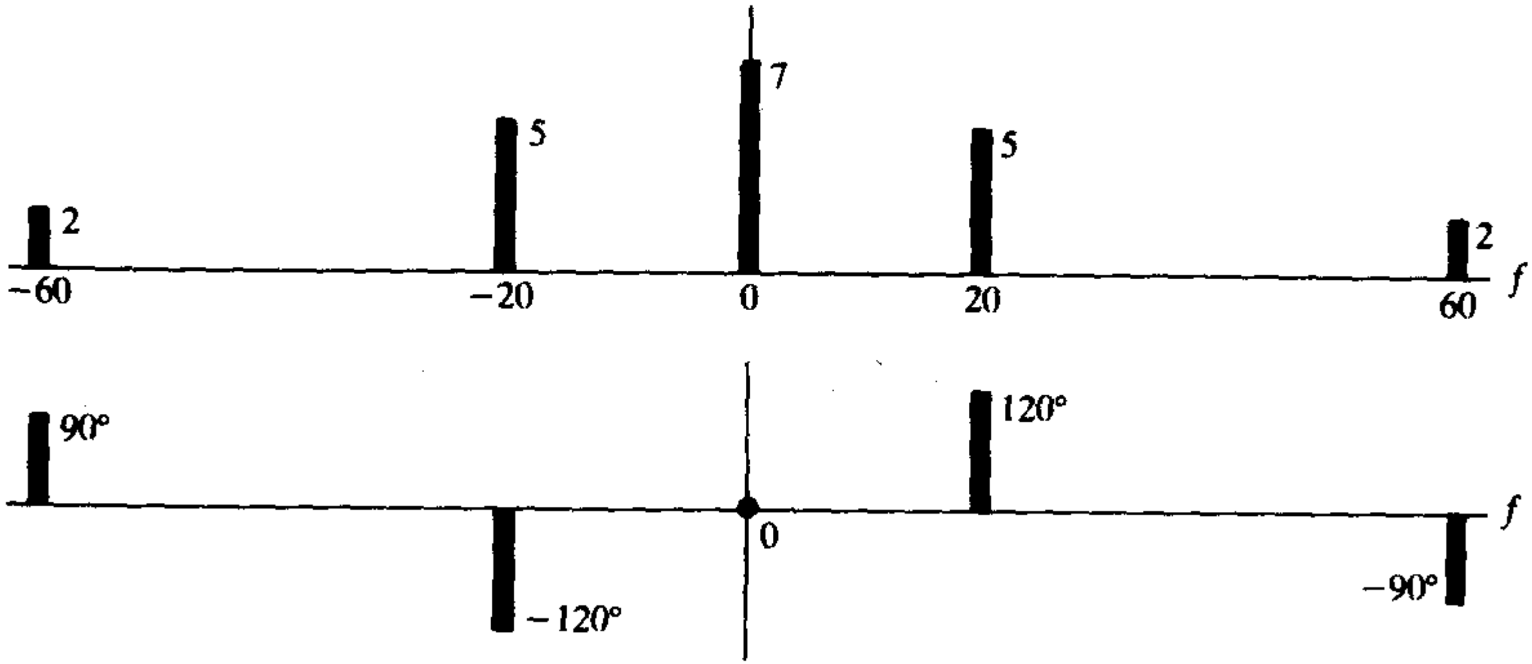
$$e^{-j\omega t} = \cos\omega t - j\sin\omega t$$

Conjugate Phasors



+ve and –ve phasors appear to rotate in opposite directions.

Two Sided Frequency Spectrum



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Fourier Transforms

Transform

$$V(f) = F[v(t)] = \int_{-\infty}^{\infty} v(t) e^{-j2\pi ft} dt$$

Inverse Transform

$$v(t) = F^{-1}[V(f)] = \int_{-\infty}^{\infty} V(f) e^{j2\pi ft} df$$

Some Fourier Transform Theorems

Time shift theorem

$$F\left[v\left(t-t_d\right)\right]=V\left(f\right)e^{-j\omega t_d}$$

Frequency translation theorem

$$F\left[v\left(t\right)e^{-j\omega_c t}\right]=V\left(f-f_c\right)$$

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Sampling

- Let the sampled signal be $x_s(t) = x(t)s(t)$ where $s(t)$ is a sampling function with period f_s.
- $s(t)$ can be written as a Fourier series.

$$s(t) = c_0 + \sum_{n=1}^{\infty} 2c_n \cos n\omega_s t$$

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Sampling

- x_s can then be written as an expansion:

$$x_s(t) = c_0 x(t) + 2c_1 x(t)\cos\omega_s t$$
$$+ 2c_2 x(t)\cos 2\omega_s t + \cdots$$

Sampling

Using trigonometic identity

$$A\cos\left(wt\right) = \frac{A}{2}e^{j\omega t} + \frac{A}{2}e^{-j\omega t}$$

and the frequency translation theorem

$$F\left[v(t)e^{j\omega_c t}\right] = V\left(f - f_c\right)$$

we can work out the sampled spectrum

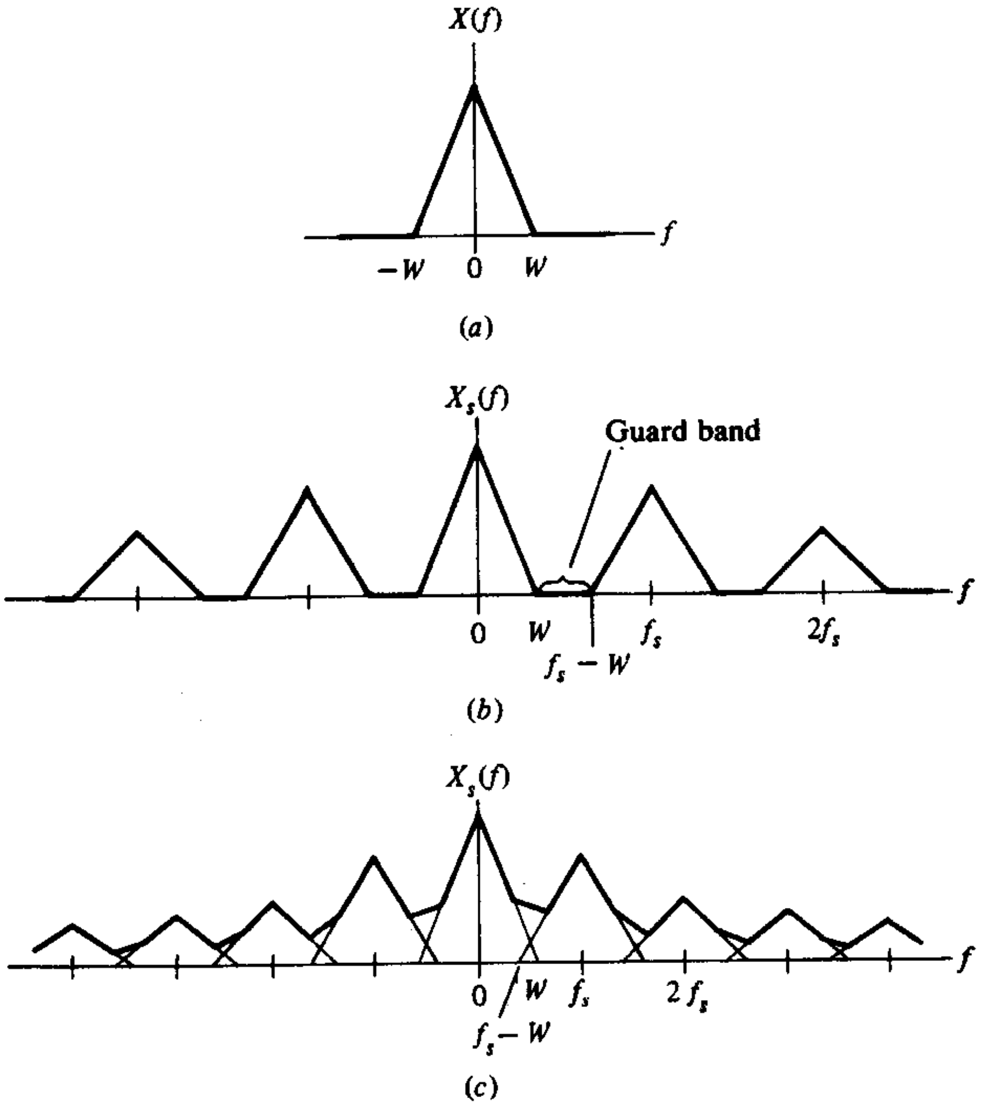
Sampling

- The Fourier transform of the sampled signal is:

$$X_s = c_0 X(f) + c_1[X(f - f_s) + X(f + f_s)] \\ + c_2[X(f - 2f_s) + X(f + 2f_s)] + \cdots$$

- So the sampled spectrum contains higher frequency copies of the original signal. Aliasing occurs when the copies overlap.

Sampled spectra



(a) Original signal spectrum

(b) Correctly sampled signal spectrum

(c) Undersampled signal spectrum exhibiting aliasing

Spectral Analysis

Take a vector of N data points $\mathbf{y} = [y_1 \; y_2 \ldots y_N]$; we call this data set a time series because transform methods are so often used in signal analysis. The data set is evenly spaced in time, so $t_{j+1} = \tau j$, where τ is the sampling interval, that is, the time increment between data points, and $j = 0, \ldots N\text{-}1$.

We define the vector $\boldsymbol{Y}$*, the discrete Fourier transform of* $\mathbf{y}$*, as*

$$Y_{k+1} = \frac{1}{N}\sum_{j=0}^{N-1} y_{j+1} e^{2\pi ijk/N}$$

where $i = \sqrt{-1}$ *and* $k = 0, \ldots N - 1$. The inverse transform is

$$y_{k+1} = \sum_{j=0}^{N-1} Y_{j+1} e^{-2\pi ijk/N}$$

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Discrete Fourier Transform

Each point Y_{k+1} of the transform has an associated frequency,

$$f_{k+1} = \frac{k}{\tau N} \quad \text{where} \quad 0 \le k \le N-1$$

The lowest (nonzero) frequency is $f_2 = 1/\tau N = 1/T$, where T is the length of the time series. To measure very low frequencies, we need to analyze long time series.

The highest frequency is $f_N = (N\text{-}1)/\ \tau N \approx 1/\tau$, so to measure very high frequencies we need to use a short sampling rate.

What is the resolution of the frequency spectrum?

Discrete FT - example

- Measure at 1 kHz for 1 second
- $\tau = 0.001$ sec
- $T = 1$ sec
- $N = 1000$
- Range of DFT is 0,1,2....999 Hz
- So the resolution is 1 Hz.
- What do you change to improve resolution?

Discrete FT - example

- Choose high sampling rate: 10 kHz
- τ = 0.0001 sec, T = 1 sec, N = 10000
- FFT runs from 0,1,2,...9999 Hz
- Same resolution but wider frequency range

OR

- Choose longer sampling time: 10 seconds
- τ = 0.001 sec, T = 10 sec, N = 10000
- FFT runs from 0,0.1,0.2,...999.9 Hz
- Better resolution in the same frequency range

Fast Fourier Transform (FFT)

A **fast Fourier transform** (**FFT**) is an algorithm to compute the discrete Fourier transform (DFT) and its inverse. Fourier analysis converts time (or space) to frequency and vice versa; an FFT rapidly computes such transformations by factorizing the DFT matrix into a product of sparse (mostly zero) factors.

FFT vs DFT

The use of the FFT redudes the number of calculations required to compute the DFT transform from $O(N^2)$ to $O(N \log_2 N)$.

As a result, fast Fourier transforms are widely used for many applications in engineering, science, and mathematics.

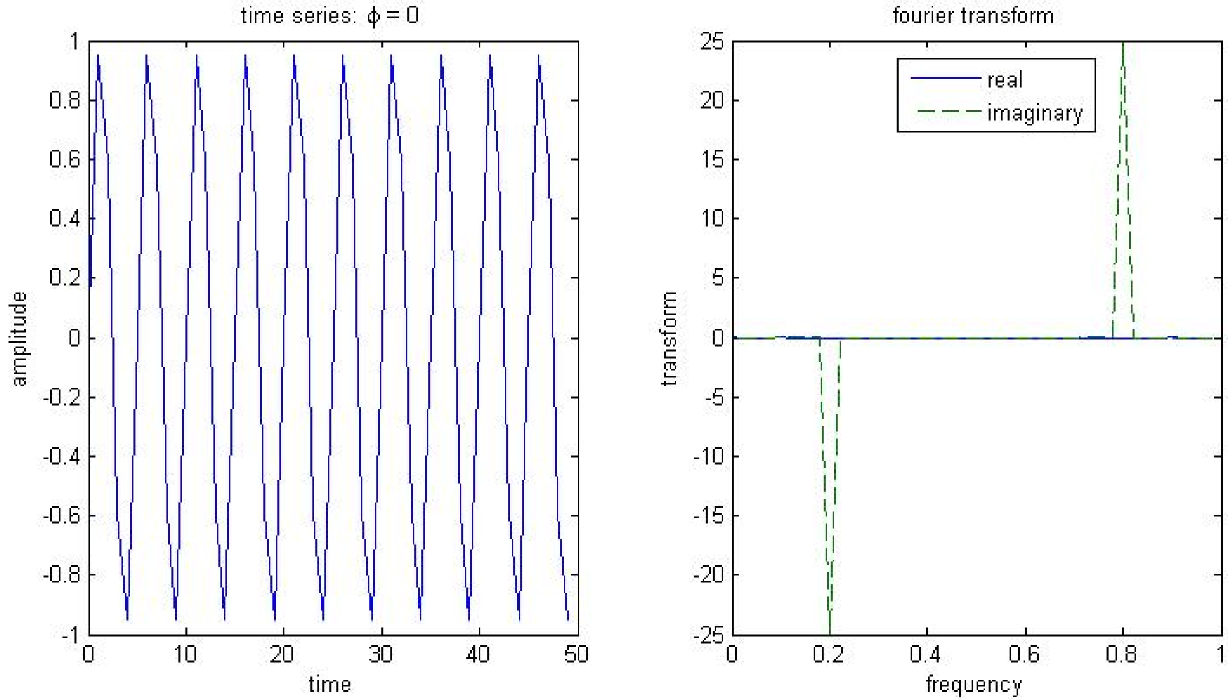
FFT Example

As an example, consider a time series constructed as follows:

$$y_{j+1} = \sin\left(2\pi f_S\, j\tau + \phi_S\right)$$

This signal is a sine wave of frequency f_S and phase ϕ_S . Note that although **y** is real, **Y** is complex, so we separately consider its real and imaginary parts. Note that the sampling frequency is implicitly 1 Hz and so the Nyquist frequency is 0.5 Hz.

FFT Example



Time series and FFT for n = 50, f_s = 0.2 and phase ϕ_s = 0.

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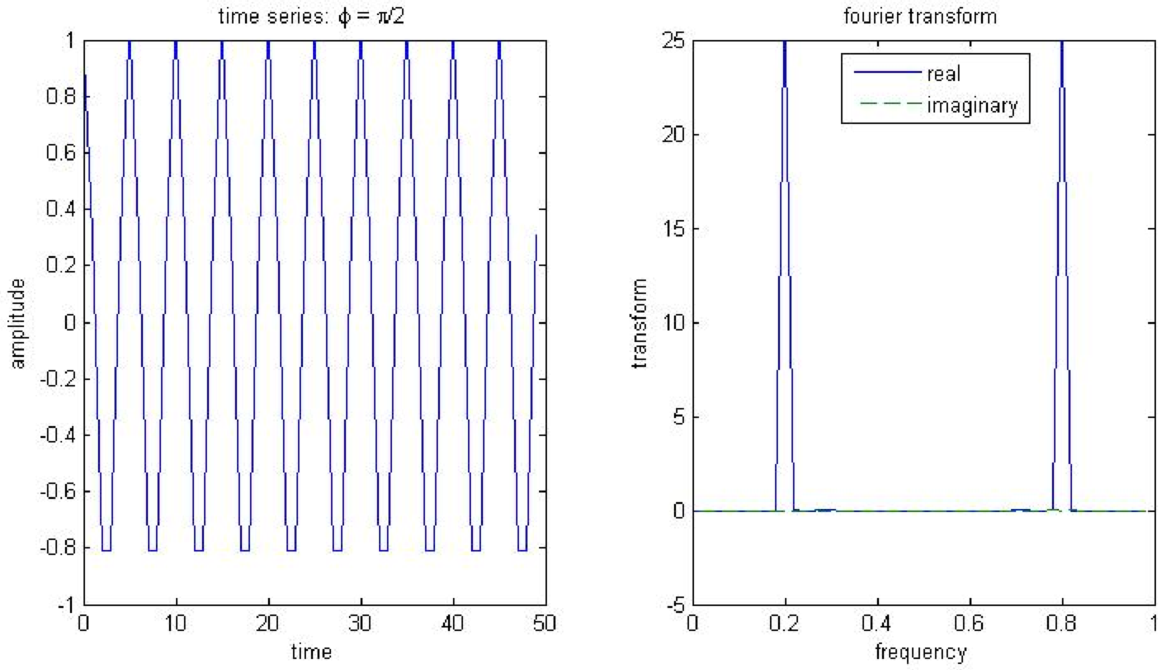
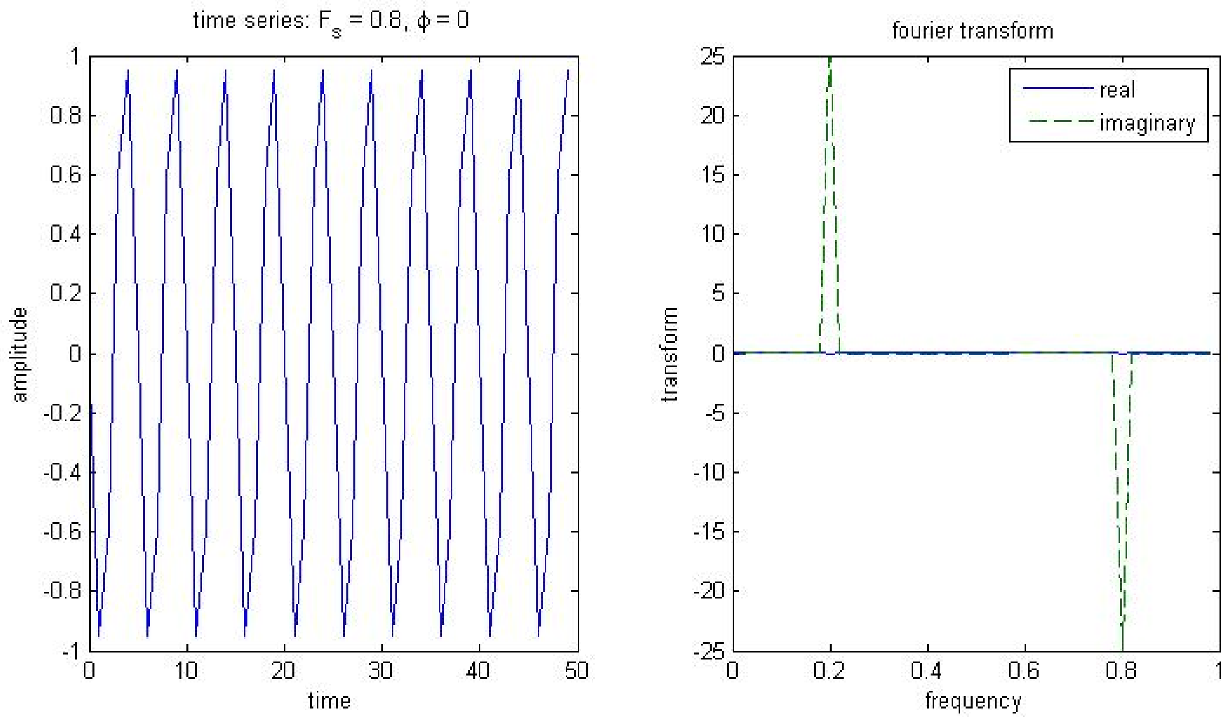
FFT Example



Time series and FFT for n = 50, f_s = 0.2 and phase $\phi_s = \pi/2$

FFT of Aliased Signal



Time series and FFT for n = 50, $f_s = 0.8$ and phase $\phi_s = 0$.
(Compare with earlier FFT for $f_s = 0.2$.)

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Reconstructed Signal and Alias

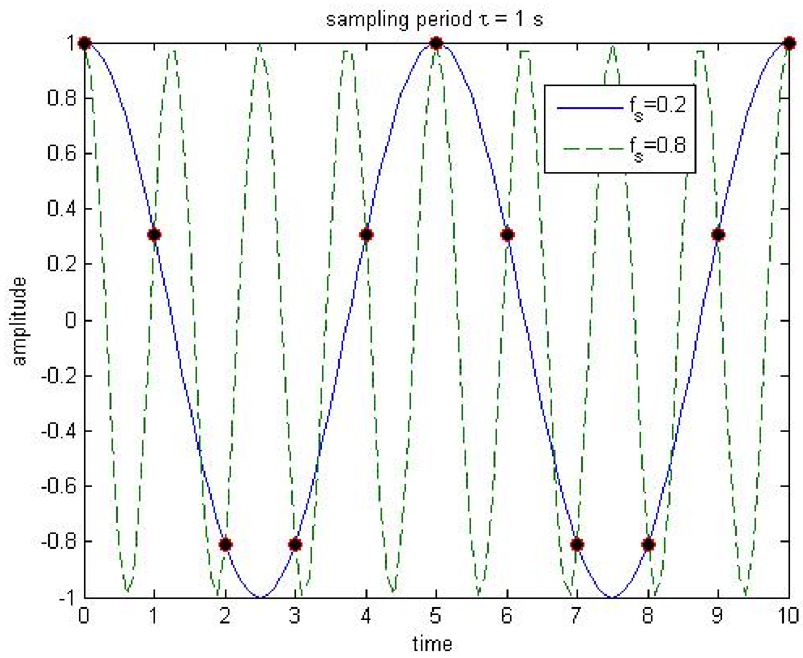

Illustration of aliasing. The two sine waves have $f_s = 0.2$ and $f_s = 0.8$; The former is shifted by $\phi_s = \pi$. When the sampling interval is $\tau = 1$ the two data sets (circles) are identical.

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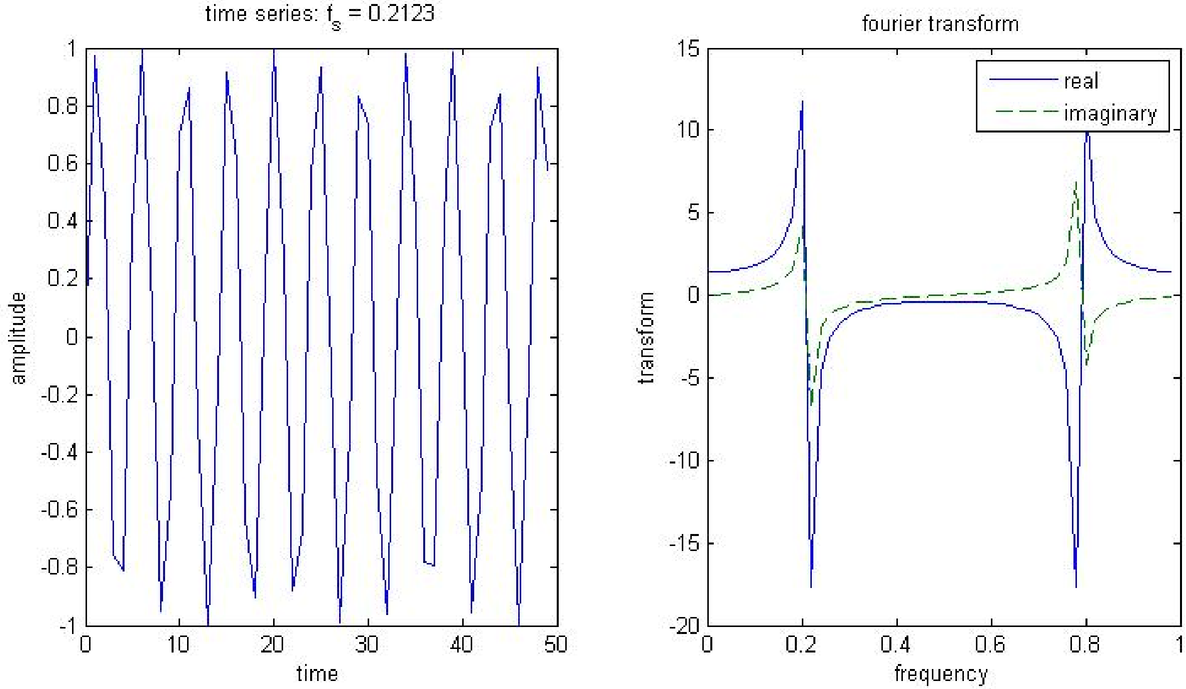
FFT Example: Spectral Leakage



Time series and FFT for n = 50, f_s = 0.2123 and phase ϕ_s = 0.

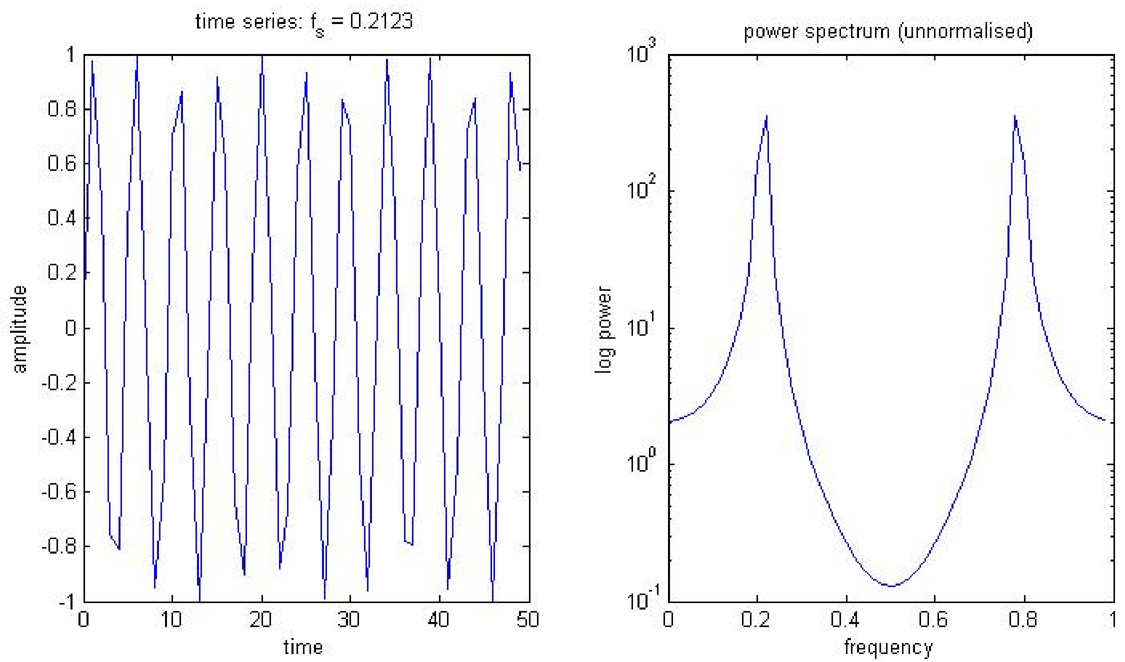
Power Spectrum

The power spectral density of the signal is given by the square of the modulus of the Fourier spectrum.

$$S_{yy} = |Y(f)|^2 = Y^*(f)Y(f)$$

This tells us where the power lies in the frequency spectrum. For measurement purposes, this is often the most useful information.

FFT Example: Power Spectrum showing effects of Spectral Leakage



Time Series and Power Spectrum for n = 50, f_s = 0.2123 and phase ϕ_s = 0, compare with earlier FFT.

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Spectral Leakage

- The Fourier Theorem applies to periodic functions.
- The FFT treats our signal sample as one period of a periodic function.
- But in this last case the beginning and end of the sample don't join up smoothly
- The discontinuity at the ends introduces extra frequencies into the spectrum.

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Spectral Leakage and Windowing

- Real data samples will always have discontinuities at the ends and so will exhibit spectral leakage.
- We cant just cut off the problematic ends as we would still have two ends!
- Instead we apply a sampling *window* function which gradually reduces the amplitude of the signal near the ends of the data set and forces them to join up.

Windowing

- Need window to eliminate spectral leakage.
- Many choices:
  - Hanning: Good general purpose choice, often start here.
  - Flat Top: choose where amplitude measurements are important.
  - Kaiser Bessel: Good dynamic range, separate similar tones of widely differing amplitude.
  - etc.